2021 Galois Contest

April 2021
(in North America and South America)

April 2021
(outside of North America and South America)

Solutions
1. (a) Substituting $a = 5$ and $b = 1$, we get $5\triangle 1 = 5(2 \times 1 + 4) = 5(6) = 30$.

(b) If $k\triangle 2 = 24$, then $k(2 \times 2 + 4) = 24$ or $8k = 24$, and so $k = 3$.

(c) Solving the given equation for $p$, we get

\begin{align*}
    p\triangle 3 &= 3\triangle p \\
    p(2 \times 3 + 4) &= 3(2p + 4) \\
    p(10) &= 6p + 12 \\
    10p - 6p &= 12 \\
    4p &= 12 \\
    p &= 3
\end{align*}

The only value of $p$ for which $p\triangle 3 = 3\triangle p$ is $p = 3$.

(d) Simplifying the given equation, we get

\begin{align*}
    m\triangle (m + 1) &= 0 \\
    m(2(m + 1) + 4) &= 0 \\
    m(2m + 2 + 4) &= 0 \\
    m(2m + 6) &= 0
\end{align*}

Thus, $m = 0$ or $2m + 6 = 0$ which gives $m = -3$.

The values of $m$ for which $m\triangle (m + 1) = 0$ are $m = 0$ and $m = -3$.

(Substituting each of these values of $m$, we may check that $0\triangle 1 = 0(2 \times 1 + 4) = 0(6) = 0$, and that $(-3)\triangle (-2) = -3(2 \times (-2) + 4) = -3(0) = 0$.)

2. (a) Team $P$ played 27 games which included 10 wins and 14 losses.

Thus, Team $P$ had $27 - 10 - 14 = 3$ ties at the end of the season.

(b) Team $Q$ had 2 more wins than Team $P$, or $10 + 2 = 12$ wins.

Team $Q$ had 4 fewer losses than Team $P$, or $14 - 4 = 10$ losses.

Since Team $Q$ played 27 games, they had $27 - 12 - 10 = 5$ ties.

At the end of the season, Team $Q$ had a total of $(2 \times 12) + (0 \times 10) + (1 \times 5)$ or 29 points.

(c) **Solution 1**

Assume that Team $R$ finished the season with exactly 6 ties.

Since 6 ties contribute 6 points to their points total, then Team $R$ earned the remaining $25 - 6 = 19$ points as a result of their wins.

However, each win contributes 2 points to the total, and thus it is not possible to earn an odd number of points from wins.

Therefore, Team $R$ could not have finished the season with exactly 6 ties.

**Solution 2**

Assume that Team $R$ finished the season with exactly $w$ wins.

If Team $R$ finished with exactly 6 ties, then they finished the season with a total of $(2 \times w) + (1 \times 6)$ or $2w + 6 = 2(w + 3)$ points (they earn 0 points for losses).

Since $w$ is an integer, then $w + 3$ is an integer and so $2(w + 3)$ is an even integer.

However, this is not possible since Team $R$ finished the season with 25 points, an odd number of points.

Therefore, Team $R$ could not have finished the season with exactly 6 ties.
(d) **Solution 1**

Let the number of losses that Team $S$ had at the end of the season be $\ell$.

Team $S$ had 4 more wins than losses and thus finished the season with $\ell + 4$ wins.

Since Team $S$ played 27 games, then each of their remaining $27 - \ell - (\ell + 4) = 23 - 2\ell$ games resulted in a tie.

Therefore, Team $S$ finished the season with a total of $(2 \times (\ell + 4)) + (0 \times \ell) + (1 \times (23 - 2\ell))$
or $2\ell + 8 + 23 - 2\ell = 31$ points.

**Solution 2**

Each of the 4 teams played 27 games, 2 teams played in each game, and so the season finished with a total of $\frac{4 \times 27}{2} = 54$ games played.

Each of the 54 games resulted in a total of 2 points being awarded (either 2 points to a winning team and 0 to the losing team or 1 point to each of the two teams that tied).

Thus, the total points earned by all 4 teams at the end of the season was $2 \times 54 = 108$.

The table shows that Team $P$ finished with 23 points, Team $R$ had 25 points, and in part (b) we determined that Team $Q$ had 29 points at the end of the season.

Therefore, Team $S$ finished the season with $108 - 23 - 25 - 29 = 31$ points.

3. (a) **Solution 1**

We begin by drawing and labelling a diagram, as shown.

The diagonals of a rectangle intersect at the centre of the rectangle. That is, $E$ is the midpoint of $AC$. Thus, the $x$-coordinate of $E$ is the average of the $x$-coordinates of $A$ and $C$, or $\frac{0+6}{2} = 3$.

The $y$-coordinate of $E$ is the average of the $y$-coordinates of $A$ and $C$, or $\frac{0+12}{2} = 6$, and so the coordinates of $E$ are $(3, 6)$.

Consider base $AD = 6$ of $\triangle ADE$, then its height is equal to the distance from $E$ to the $x$-axis, which is 6.

The area of $\triangle ADE$ is $\frac{1}{2}(6)(6) = 18$.

**Solution 2**

The diagonals of a rectangle divide the rectangle into 4 non-overlapping triangles having equal area. (You should consider why this is true before reading on.)

Thus, the area of $\triangle ADE$ is equal to $\frac{1}{4}$ of the area of rectangle $ABCD$ or $\frac{1}{4}(6)(12) = 18$.

(b) **Solution 1**

We begin by drawing and labelling a diagram, as shown.

The area of rectangle $ABCD$ is equal to the area of trapezoid $BCDP$ plus the area of $\triangle PAD$.

Since the area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, then the area of $\triangle PAD$ is $\frac{1}{3}$ the area of $ABCD$ (and the area of trapezoid $BCDP$ is $\frac{2}{3}$ the area of $ABCD$).

The area of rectangle $ABCD$ is $6 \times 12 = 72$, and so the area of $\triangle PAD$ is $\frac{1}{3} \times 72 = 24$.

The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$, and so $3p = 24$ or $p = 8$.

**Solution 2**

Point $P$ has coordinates $(0, p)$ and so $AP = p$ and $BP = 12 - p$.

The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$.

The area of trapezoid $BCDP$ is $\frac{1}{2}(BC)(BP + CD) = \frac{1}{2}(6)(12 - p + 12) = 3(24 - p)$.
The area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, and so $3(24 - p) = 2(3p)$ or $24 - p = 2p$, and so $3p = 24$ or $p = 8$.

(c) The area of rectangle $ABCD$ is $6 \times 12 = 72$.

The sum of the areas of the two trapezoids is equal to the area of rectangle $ABCD$.

Since the ratio of the areas of these two trapezoids is $5 : 3$, then the areas of the two trapezoids are $\frac{5}{8} \times 72 = 45$ and $\frac{3}{8} \times 72 = 27$.

(We may check that $45 : 27 = 5 : 3$ and $45 + 27 = 72$.)

Let $\ell$ be the line that passes through $U$, $V$ and $W$.

Begin by assuming $\ell$ does not pass through a vertex of $ABCD$. In this case, $\ell$ either intersects opposite sides of $ABCD$, or it intersects adjacent sides of $ABCD$.

If $\ell$ intersects opposite sides of $ABCD$, then $\ell$ divides $ABCD$ into two trapezoids, as required.

If $\ell$ intersects adjacent sides of $ABCD$, then $\ell$ divides $ABCD$ into a triangle and a pentagon. This is not possible.

Assume $\ell$ passes through at least one vertex of $ABCD$.

In this case, $\ell$ divides $ABCD$ into two figures, at least one of which is a triangle. This is not also possible.

Thus, $\ell$ intersects opposite sides of $ABCD$ and does not pass through $A$, $B$, $C$, or $D$.

That is, line $\ell$ can intersect opposite sides of $ABCD$ in the two different ways shown below.

In each case, since $\ell$ is a straight line passing through $U$, $V$ and $W$, then the slope of $UV$ is equal to the slope of $VW$.

That is,

\[
\frac{4 - u}{2 - 0} = \frac{w - 4}{6 - 2} \quad 4(4 - u) = 2(w - 4) \quad 2(4 - u) = w - 4 \quad 8 - 2u = w - 4 \quad w = 12 - 2u
\]
Case 1: Line \( \ell \) intersects sides \( AB \) and \( CD \).
That is, \( U \) lies between \( A \) and \( B \), and \( W \) lies between \( C \) and \( D \).

In this case, \( 0 < u < 12 \), \( 0 < w < 12 \), \( AU = u \), and \( DW = w \).

The area of trapezoid \( ADWU \) is
\[
\frac{1}{2}(AD)(DW + AU) = \frac{1}{2}(6)(w + u) = 3(w + u).
\]

Since \( w = 12 - 2u \), the area of trapezoid \( ADWU \) becomes \( 3(12 - u) \).

We consider each of two possibilities: the area of trapezoid \( ADWU \) is equal to 27, or the area is equal to 45.

If the area of trapezoid \( ADWU \) is equal to 27, then
\[
3(12 - u) = 27
\]
\[
12 - u = 9
\]
\[
u = 3
\]

Substituting \( u = 3 \) into \( w = 12 - 2u \), we get \( w = 12 - 6 = 6 \).

The Case 1 conditions that \( 0 < u < 12 \) and \( 0 < w < 12 \) are satisfied and thus the ratio of the areas of the two trapezoids is 5 : 3 for the pair of points \( U(0, 3) \) and \( W(6, 6) \).

If the area of trapezoid \( ADWU \) is equal to 45, then
\[
3(12 - u) = 45
\]
\[
12 - u = 15
\]
\[
u = -3
\]

Here, the condition that \( 0 < u < 12 \) is not satisfied and so there is no pair of points \( U \) and \( W \) for which the ratio of the areas of the two trapezoids is 5 : 3.

Case 2: Line \( \ell \) intersects sides \( AD \) and \( BC \).
That is, \( U \) lies on \( AB \) extended, outside of side \( AB \), and \( W \) lies on \( CD \) extended, outside of side \( CD \).

We begin by drawing and labelling a diagram, including \( E(e, 0) \) and \( F(f, 12) \), the points where \( \ell \) intersects sides \( AD \) and \( BC \) respectively, as shown.

In this case, \( u < 0 \) and \( w > 12 \) (as in the diagram shown), or \( u > 12 \) and \( w < 0 \) (when \( U \) lies above \( B \) and \( W \) lies below \( D \)).

We note that what follows is true for each of these two cases, and thus we need not consider them separately.

In this case, we require that \( 0 < e < 6 \), \( 0 < f < 6 \), and so we get \( BF = f \) and \( AE = e \).

The area of trapezoid \( BFEA \) is
\[
\frac{1}{2}(AB)(BF + AE) = \frac{1}{2}(12)(f + e) = 6(f + e).
\]

Further, since \( \ell \) is a straight line passing through \( E \), \( V \) and \( F \), then the slope of \( EV \) is equal to the slope of \( FV \).
That is,

\[
\begin{align*}
\frac{4 - 0}{2 - e} &= \frac{12 - 4}{f - 2} \\
\frac{4}{2 - e} &= \frac{8}{f - 2} \\
4(f - 2) &= 8(2 - e) \\
f - 2 &= 2(2 - e) \\
f &= 6 - 2e
\end{align*}
\]

Since \( f = 6 - 2e \), the area of trapezoid \( BF EA \) becomes \( 6(6 - e) \).

We consider each of two possibilities: the area of trapezoid \( BF EA \) is equal to 27, or the area is equal to 45.

If the area of trapezoid \( BF EA \) is equal to 27, then

\[
\begin{align*}
6(6 - e) &= 27 \\
6 - e &= \frac{9}{2} \\
e &= \frac{3}{2}
\end{align*}
\]

Substituting \( e = \frac{3}{2} \) into \( f = 6 - 2e \), we get \( f = 3 \), and these values satisfy the Case 2 conditions \( 0 < e < 6 \) and \( 0 < f < 6 \).

Here, we get \( E(\frac{3}{2}, 0) \) and \( F(3, 12) \) and use these points to determine \( U \) and \( W \).

The slope of \( FV \) is \( \frac{12 - 4}{3 - 2} = 8 \) and so the slope of \( WV \) is also 8, which gives \( \frac{w - 4}{4} = 8 \), and solving we get \( w = 36 \).

Similarly, the slope of \( VU \) is also 8, which gives \( \frac{4 - u}{2} = 8 \), and solving we get \( u = -12 \).

We note that \( w = 36 \) and \( u = -12 \) satisfy the conditions \( w > 12 \) and \( u < 0 \) and so the ratio of the areas of the two trapezoids is \( 5 : 3 \) for the points \( U(0, -12) \) and \( W(6, 36) \).

If the area of trapezoid \( BF EA \) is equal to 45, then

\[
\begin{align*}
6(6 - e) &= 45 \\
6 - e &= \frac{15}{2} \\
e &= -\frac{3}{2}
\end{align*}
\]

Here, the condition that \( 0 < e < 6 \) is not satisfied and so there is no pair of points \( E \) and \( F \) and thus no pair of points \( U \) and \( W \) for which the ratio of the areas of the two trapezoids is \( 5 : 3 \).

Thus, there are two pairs of points \( U \) and \( W \) for which the ratio of the areas of the two trapezoids is \( 5 : 3 \). These are \( U(0, 3), W(6, 6) \), and \( U(0, -12), W(6, 36) \).
4. (a) When $x = 6$, $\frac{5}{x} + \frac{14}{y} = 2$ becomes $\frac{5}{6} + \frac{14}{y} = 2$ and so $\frac{14}{y} = 2 - \frac{5}{6} = \frac{7}{6}$, which gives $y = 12$.

(b) Solution 1

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

\[
\frac{4}{x} + \frac{5}{y} = 1
\]

\[
\frac{4}{x}(xy) + \frac{5}{y}(xy) = 1(xy) \quad \text{(since } xy \neq 0) \]

\[
4y + 5x = xy
\]

\[
x(y - 5) - 4y = 0
\]

\[
x(y - 5) - 4(y - 5) = 20
\]

\[
(x - 4)(y - 5) = 20
\]

Since $x$ and $y$ are positive integers, then $x - 4$ and $y - 5$ are integers and thus are a factor pair of 20.

Since $y > 0$, then $y - 5 > -5$.

The factors of 20 which are greater than $-5$ are: $-4, -2, -1, 1, 2, 4, 5, 10,$ and $20$.

If $y - 5$ is equal to $-4$, then $x - 4 = -5$ (since $(-5)(-4) = 20$), and so $x = -1$.

This is not possible since $x$ is a positive integer.

Similarly, $y - 5$ cannot equal $-2$ or $-1$ (since each gives $x < 0$), and so $y - 5$ is a positive factor of 20.

In the table below, we determine the values of $x$ and $y$ corresponding to each of the positive factor pairs of 20.

<table>
<thead>
<tr>
<th>Factor Pair</th>
<th>$x - 4$</th>
<th>$y - 5$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 20</td>
<td>1</td>
<td>20</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>20 and 1</td>
<td>20</td>
<td>1</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td>2 and 10</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>10 and 2</td>
<td>10</td>
<td>2</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>4 and 5</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>5 and 4</td>
<td>5</td>
<td>4</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

Thus, the ordered pairs of positive integers $(x, y)$ that are solutions to the given equation are $(5, 25)$, $(24, 6)$, $(6, 15)$, $(14, 7)$, $(8, 10)$, and $(9, 9)$. 
Solution 2

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

\[
\frac{4}{x} + \frac{5}{y} = 1
\]

\[
\frac{4}{x}(xy) + \frac{5}{y}(xy) = 1(xy) \quad \text{(since } xy \neq 0)\]

\[
4y + 5x = xy
\]

\[
x(y - 5) = 4y.
\]

\[
x = \frac{4y}{y - 5} \quad (y \neq 5)
\]

\[
x = \frac{4y - 20 + 20}{y - 5}
\]

\[
x = \frac{4(y - 5) + 20}{y - 5}
\]

\[
x = 4 + \frac{20}{y - 5}
\]

Since $x$ and $y$ are positive integers, then $y - 5$ is a divisor of 20.

Since $y > 0$, then $y - 5 > -5$.

The divisors of 20 which are greater than $-5$ are: $-4, -2, -1, 1, 2, 4, 5, 10,$ and $20$.

If $y - 5$ is equal to $-4$, then $x = 4 + \frac{20}{-4} = -1$, which is not possible since $x$ is a positive integer.

Similarly, $y - 5$ cannot equal $-2$ or $-1$ (since each gives $x < 0$), and so $y - 5$ is a positive divisor of 20.

In the table below, we determine the values of $y$ and $x$ corresponding to each of the positive divisors of 20.

<table>
<thead>
<tr>
<th>$y - 5$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>$x$</td>
<td>24</td>
<td>14</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

Thus, the ordered pairs of positive integers $(x, y)$ that are solutions to the given equation are $(24, 6), (14, 7), (9, 9), (8, 10), (6, 15),$ and $(5, 25)$. 
(c) Solution 1

Since \( x \geq 1 \) and \( y \geq 1 \), then \( \frac{16}{x} + \frac{25}{y} \leq 16 + 25 = 41 \), and so \( 5 \leq p \leq 41 \). That is, the possible prime numbers \( p \) come from the list 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, and 41.

Since \( x \) and \( y \) are positive integers, we obtain the following equivalent equations,

\[
\frac{16}{x} + \frac{25}{y} = p \\
\frac{16}{x}(xy) + \frac{25}{y}(xy) = p(xy) \quad \text{(since } xy \neq 0) \\
16y + 25x = pxy \\
p\times - 25x - 16y = 0 \\
p^2xy - 25px - 16py = 0 \quad \text{(since } p > 0) \\
p\times(p\times - 25) - 16py = 0 \\
p\times(p\times - 25) - 16(p\times - 25) = 400 \\
(px - 16)(py - 25) = 400
\]

Since \( p, x \) and \( y \) are positive integers, then \( px - 16 \) and \( py - 25 \) are integers and thus are a factor pair of 400.

Since \( p \geq 5 \) and \( x \geq 1 \), then \( px \geq 5 \), and so \( px - 16 \geq 5 - 16 \) or \( px - 16 \geq -11 \).

The factors of 400 which are greater than or equal to \(-11\), and are less than 0, are: \(-1, -2, -4, -5, -8, \text{ and } -10\).

If \( px - 16 = -1 \), then \( py - 25 = -400 \).

In this case, we get \( py = -375 \) which is not possible since both \( p \) and \( y \) are positive.

We can similarly show that \( px - 16 \) cannot equal \(-2, -4, -5, -8, \text{ and } -10\) (since each gives \( py < 0 \)) and so \( px - 16 \) is a positive factor of 400 and thus \( py - 25 \) is also.

In the table below, we determine possible values of \( p \) corresponding to each of the positive factor pairs of 400.

Recall from earlier that we only need to consider possible values of \( p \) for which \( 5 \leq p \leq 41 \).

<table>
<thead>
<tr>
<th>( px - 16 )</th>
<th>( py - 25 )</th>
<th>( px )</th>
<th>( py )</th>
<th>New common prime factor of the integers ( px ) and ( py )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>400</td>
<td>17</td>
<td>425 = 17 \times 25</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>18</td>
<td>225</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>20 = 5 \times 4</td>
<td>125 = 5 \times 25</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td>21 = 7 \times 3</td>
<td>105 = 7 \times 15</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>24</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>26 = 13 \times 2</td>
<td>65 = 13 \times 5</td>
<td>13</td>
</tr>
<tr>
<td>16</td>
<td>25</td>
<td>32</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>36</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>16</td>
<td>41</td>
<td>41</td>
<td>41</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>56</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>8</td>
<td>66 = 11 \times 6</td>
<td>33 = 11 \times 3</td>
<td>11</td>
</tr>
<tr>
<td>80</td>
<td>5</td>
<td>96</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>4</td>
<td>116 = 29 \times 4</td>
<td>29</td>
<td>29</td>
</tr>
<tr>
<td>200</td>
<td>2</td>
<td>216</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>1</td>
<td>416</td>
<td>26</td>
<td></td>
</tr>
</tbody>
</table>
The values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the given equation are 5, 7, 11, 13, 17, 29, and 41. We may check, for example, that when $(x, y) = (6, 3)$ we get,

$$\frac{16}{x} + \frac{25}{y} = \frac{16}{6} + \frac{25}{3} = \frac{16}{6} + \frac{50}{6} = \frac{66}{6} = 11$$

as given in the table above.

**Solution 2**

Since $x \geq 1$ and $y \geq 1$, then $\frac{16}{x} + \frac{25}{y} \leq 16 + 25 = 41$, and so $5 \leq p \leq 41$. That is, the possible prime numbers $p$ come from the list 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, and 41.

When $x$ is a positive divisor of 16, $\frac{16}{x}$ is a positive integer.

Specifically, when $x = 1, 2, 4, 8, 16$, the values of $\frac{16}{x}$ are 16, 8, 4, 2, 1, respectively.

Similarly, when $y$ is a positive divisor of 25, $\frac{25}{y}$ is a positive integer.

Specifically, when $y = 1, 5, 25$, the values of $\frac{25}{y}$ are 25, 5, 1, respectively.

We may use this observation to determine some values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the equation.

We summarize these solutions in the table below.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x$</th>
<th>$y$</th>
<th>$\frac{16}{x} + \frac{25}{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>25</td>
<td>$\frac{16}{4} + \frac{25}{25} = 4 + 1$</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>5</td>
<td>$\frac{16}{8} + \frac{25}{5} = 2 + 5$</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>5</td>
<td>$\frac{16}{2} + \frac{25}{5} = 8 + 5$</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>25</td>
<td>$\frac{16}{1} + \frac{25}{25} = 16 + 1$</td>
</tr>
<tr>
<td>29</td>
<td>4</td>
<td>1</td>
<td>$\frac{16}{4} + \frac{25}{1} = 4 + 25$</td>
</tr>
<tr>
<td>41</td>
<td>1</td>
<td>1</td>
<td>$\frac{16}{1} + \frac{25}{1} = 16 + 25$</td>
</tr>
</tbody>
</table>

From our previous list of possible values of $p$, we have only 11, 19, 23, 31, and 37 remaining to consider.
Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

$$\frac{16}{x} + \frac{25}{y} = p$$

$$\frac{16}{x}(xy) + \frac{25}{y}(xy) = p(xy) \quad \text{(since } xy \neq 0)$$

$$16y + 25x = pxy$$

$$pxy - 25x = 16y$$

$$x(py - 25) = 16y$$

$$x = \frac{16y}{py - 25} \quad \text{($p \geq 11$ and so no multiple of $p$ can equal 25)}$$

Since $x > 0$ and $16y > 0$ and $x = \frac{16y}{py - 25}$, then $py - 25 > 0$ and so $py > 25$.

Further, $x$ is an integer and so $x \geq 1$, which gives $\frac{16y}{py - 25} \geq 1$.

Simplifying, we get $16y \geq py - 25$ or $py - 16y \leq 25$, and so $y \leq \frac{25}{p - 16}$ when $p > 16$.

We may use this inequality to determine restrictions on $y$ given each of the remaining possible values of $p$ which are greater than 16, namely 37, 31, 23, and 19.

For example if $p = 37$, then $y \leq \frac{25}{37 - 16}$ or $y \leq \frac{25}{21}$, and so $y = 1$. However, when $p = 37$ and $y = 1$, we get $x = \frac{16(1)}{37(1) - 25} = \frac{16}{12}$ which is not an integer, and thus $p \neq 37$.

We summarize similar work for $p = 31, 23, 19$ in the table below noting that when $y = 1$ and $p = 23$ or $p = 19$ we get $py < 25$ (earlier we showed $py > 25$), and thus we need not consider these two cases.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$y \leq \frac{25}{p-16}$</th>
<th>Possible integer values of $y$</th>
<th>Corresponding values of $x = \frac{16y}{py-25}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$y \leq \frac{25}{31-16} = \frac{25}{15}$</td>
<td>$y = 1$</td>
<td>$x = \frac{16}{6}$</td>
</tr>
<tr>
<td>23</td>
<td>$y \leq \frac{25}{23-16} = \frac{25}{7}$</td>
<td>$y = 2, 3$</td>
<td>$x = \frac{32}{21}, \frac{48}{44}$</td>
</tr>
<tr>
<td>19</td>
<td>$y \leq \frac{25}{19-16} = \frac{25}{3}$</td>
<td>$y = 2, 3, 4, 5, 6, 7, 8$</td>
<td>$x = \frac{32}{13}, \frac{48}{32}, \frac{64}{51}, \frac{80}{70}, \frac{96}{89}, \frac{112}{108}, \frac{128}{127}$</td>
</tr>
</tbody>
</table>

Since there are no integer values of $x$, then $p \neq 19, 23, 31, 37$.

The final remaining value to check is $p = 11$.

As noted earlier, $py > 25$ and so when $p = 11$, we get $y > \frac{25}{11}$ or $y \geq 3$ (since $y$ is an integer).

Trying $y = 3$, we get $x = \frac{16(3)}{11(3) - 25} = \frac{48}{8} = 6$ and so when $p = 11$, $(x, y) = (6, 3)$ is a solution to the equation.

Summarizing, the values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the equation are 5, 7, 11, 13, 17, 29, and 41.