2023 Canadian Senior Mathematics Contest

Wednesday, November 15, 2023
(in North America and South America)

Thursday, November 16, 2023
(outside of North America and South America)

Solutions
Part A

1. Since $p + q$ is odd (because 31 is odd) and $p$ and $q$ are integers, then one of $p$ and $q$ is even and the other is odd. (If both were even or both were odd, their sum would be even.)
   Since $p$ and $q$ are both prime numbers and one of them is even, then one of them must be 2, since 2 is the only even prime number.
   Since their sum is 31, the second number must be 29, which is prime.
   Therefore, $pq = 2 \cdot 29 = 58$.

   Answer: 58

2. The integers between 100 to 999, inclusive, are exactly the three-digit positive integers.
   Consider three-digit integers of the form $abc$ where the digit $a$ is even, the digit $b$ is even, and the digit $c$ is odd.
   There are 4 possibilities for $a$: 2, 4, 6, 8. (We note that $a$ cannot equal 0.)
   There are 5 possibilities for $b$: 0, 2, 4, 6, 8.
   There are 5 possibilities for $c$: 1, 3, 5, 7, 9.
   Each choice of digits from these lists gives a distinct integer that satisfies the conditions.
   Therefore, the number of such integers is $4 \cdot 5 \cdot 5 = 100$.

   Answer: 100

3. Solution 1
   Since the distance from $(0, 0)$ to $(x, y)$ is 17, then $x^2 + y^2 = 17^2$.
   Since the distance from $(16, 0)$ to $(x, y)$ is 17, then $(x - 16)^2 + y^2 = 17^2$.
   Subtracting the second of these equations from the first, we obtain $x^2 - (x - 16)^2 = 0$ which gives $x^2 - (x^2 - 32x + 256) = 0$ and so $32x = 256$ or $x = 8$.
   Since $x = 8$ and $x^2 + y^2 = 17^2$, then $64 + y^2 = 289$ which gives $y^2 = 225$, from which we get $y = 15$ or $y = -15$.
   Therefore, the two possible pairs of coordinates for $P$ are $(8, 15)$ and $(8, -15)$.

Solution 2
   The point $P$ is equidistant from $O$ and $A$ since $OP = PA = 17$.
   Suppose that $M$ is the midpoint of $OA$.
   Since $O$ has coordinates $(0, 0)$ and $A$ has coordinates $(16, 0)$, then $M$ has coordinates $(8, 0)$.
   Since $OP = PA$, then $\triangle OPA$ is isosceles.
   This means that median $PM$ in $\triangle OPA$ is also an altitude; in other words, $PM$ is perpendicular to $OA$.
   Since $OA$ is horizontal, $PM$ is vertical, and so $P$ lies on the vertical line with equation $x = 8$.
   Since $OM = 8$ and $OP = 17$ and $\triangle POM$ is right-angled at $M$, then by the Pythagorean Theorem, $PM = \sqrt{OP^2 - OM^2} = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$.
   Since $PM$ is vertical and $M$ is on the $x$-axis, then $P$ is a distance of 15 units vertically from the $x$-axis.
   Since $P$ has $x$-coordinate 8 and is 15 units away from the $x$-axis, then the two possible pairs of coordinates for $P$ are $(8, 15)$ and $(8, -15)$.

Answer: $(8, 15), (8, -15)$
4. The store sold $x$ shirts for $10$ each, $y$ water bottles for $5$ each, and $z$ chocolate bars for $1$ each.
Since the total revenue was $120$, then $10x + 5y + z = 120$.
Since $z = 120 - 10x - 5y$ and each term on the right side is a multiple of $5$, then $z$ is a multiple of $5$.
Set $z = 5t$ for some integer $t > 0$.
This gives $10x + 5y + 5t = 120$. Dividing by $5$, we obtain $2x + y + t = 24$.
Since $x > 0$ and $x$ is an integer, then $x \geq 1$.
Since $y > 0$ and $t > 0$, then $y + t \geq 2$ (since $y$ and $t$ are integers).
This means that $2x = 24 - y - t \leq 22$ and so $x \leq 11$.
If $x = 1$, then $y + t = 22$. There are $21$ pairs $(y, t)$ that satisfy this equation, namely the pairs $(y, t) = (1, 21), (2, 20), (3, 19), \ldots, (20, 2), (21, 1)$.
If $x = 2$, then $y + t = 20$. There are $19$ pairs $(y, t)$ that satisfy this equation, namely the pairs $(y, t) = (1, 19), (2, 18), (3, 17), \ldots, (18, 2), (19, 1)$.
For each value of $x$ with $1 \leq x \leq 11$, we obtain $y + t = 24 - 2x$.
Since $y \geq 1$, then $t \leq 23 - 2x$.
Since $t \geq 1$, then $y \leq 23 - 2x$.
In other words, $1 \leq y \leq 23 - 2x$ and $1 \leq t \leq 23 - 2x$.
Furthermore, picking any integer $y$ satisfying $1 \leq y \leq 23 - 2x$ gives a positive value of $t$, and so there are $23 - 2x$ pairs $(y, t)$ that are solutions.
Therefore, as $x$ ranges from $1$ to $11$, there are

$$21 + 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1$$

pairs $(y, t)$, which means that there are this number of triples $(x, y, z)$.
This sum can be re-written as

$$21 + (19 + 1) + (17 + 3) + (15 + 5) + (13 + 7) + (11 + 9)$$

or $21 + 5 \cdot 20$, which means that the number of triples is $121$.

**Answer:** $121$

5. We consider $r^2 - r(p + 6) + p^2 + 5p + 6 = 0$ to be a quadratic equation in $r$ with two coefficients that depend on the variable $p$.
For this quadratic equation to have real numbers $r$ that are solutions, its discriminant, $\Delta$, must be greater than or equal to $0$. A non-negative discriminant does not guarantee integer solutions, but may help us narrow the search.
By definition,

$$\Delta = (-p - 6)^2 - 4 \cdot 1 \cdot (p^2 + 5p + 6)$$

$$= p^2 + 12p + 36 - 4p^2 - 20p - 24$$

$$= -3p^2 - 8p + 12$$

Thus, we would like to find all integer values of $p$ for which $-3p^2 - 8p + 12 \geq 0$. The set of integers $p$ that satisfy this inequality are the only possible values of $p$ which could be part of a solution pair $(r, p)$ of integers. We can visualize the left side of this inequality as a parabola opening downwards, so there will be a finite range of values of $p$ for which this is true.
By the quadratic formula, the solutions to the equation $-3p^2 - 8p + 12 = 0$ are

$$p = \frac{8 \pm \sqrt{8^2 - 4(-3)(12)}}{2(-3)} = \frac{8 \pm \sqrt{208}}{-6} \approx 1.07, -3.74$$
Since the roots of the equation \(-3p^2 - 8p + 12 = 0\) are approximately 1.07 and -3.74, then the integers \(p\) for which \(-3p^2 - 8p + 12 \geq 0\) are \(p = -3, -2, -1, 0, 1\). (These values of \(p\) are the only integers between the real solutions 1.07 and -3.74.)

It is these values of \(p\) for which there are possibly integer values of \(r\) that work.

We try them one by one:

- When \(p = 1\), the original equation becomes \(r^2 - 7r + 12 = 0\), which gives \((r - 3)(r - 4) = 0\), and so \(r = 3\) or \(r = 4\).
- When \(p = 0\), the original equation becomes \(r^2 - 6r + 6 = 0\). Using the quadratic formula, we can check that this equation does not have integer solutions.
- When \(p = -1\), the original equation becomes \(r^2 - 5r + 2 = 0\). Using the quadratic formula, we can check that this equation does not have integer solutions.
- When \(p = -2\), the original equation becomes \(r^2 - 4r = 0\), which factors as \(r(r - 4) = 0\), and so \(r = 0\) or \(r = 4\).
- When \(p = -3\), the original equation becomes \(r^2 - 3r = 0\), which factors as \(r(r - 3) = 0\), and so \(r = 0\) or \(r = 3\).

Therefore, the pairs of integers that solve the equation are

\[(r, p) = (3, 1), (4, 1), (0, -2), (4, -2), (0, -3), (3, -3)\]

**Answer:** \((3, 1), (4, 1), (0, -2), (4, -2), (0, -3), (3, -3)\)
6. We start by determining the heights above the bottom of the cube of the points of intersection of the edges of the pyramids.
For example, consider square $AFGB$ and edges $AG$ and $FP$. We call their point of intersection $X$.
We assign coordinates to the various points using the fact that the edge length of the cube is 6:
$F(0,0)$, $G(6,0)$, $B(6,6)$, $A(0,6)$, $P(6,3)$ ($P$ is the midpoint of $BG$).

![Image of a cube with points labeled]

Line segment $AG$ has slope $-1$, and so has equation $y = -x + 6$.
Line segment $FP$ has slope $\frac{3}{6} = \frac{1}{2}$ and so has equation $y = \frac{1}{2}x$.
To find the coordinates of $X$, we equate expressions in $y$ to obtain $-x + 6 = \frac{1}{2}x$ which gives $\frac{3}{2}x = 6$ or $x = 4$, and so $y = -4 + 6 = 2$.
Therefore, point $X$ is a height of 2 above square $EFGH$.
Using a similar argument, the point of intersection between $PH$ and $GC$ is 2 units above square $EFGH$.
To see why the point of intersection of $GD$ and $PE$ is also 2 units above $EFGH$, we note that rectangle $DEGB$ has a height of 6 (like square $AFGB$) and a width of $6\sqrt{2}$. As a result, we can think of obtaining rectangle $DEGB$ by stretching square $AFGB$ horizontally by a factor of $\sqrt{2}$. This horizontal stretch will not raise or lower the point of intersection between $GD$ and $PE$ and so this point is also two units above $EFGH$.
Now, imagine drawing a plane through the three points of intersection of the edges of the pyramids.
Since each of these points is 2 units above $EFGH$, this plane must be horizontal and will also intersect $BG$ 2 units above $G$, forming a square. (The points of intersection form a square because every horizontal cross-section of both pyramids is a square.) This square has side length 2 because the $x$-coordinate of $X$ was 4, which is 2 units from $BG$ in that coordinate system.
This square divides the common three-dimensional region into two square-based pyramids.
One of these pyramids points upwards and has fifth vertex $P$. This pyramid has a square base with edge length 2 and a height of $3 - 2 = 1$, since $P$ is 3 units above $G$ and the base of the pyramid is 2 units above $G$.
The other pyramid points downwards and has fifth vertex $G$. This pyramid has a square base with edge length 2 and a height of 2.
Thus, the volume of the region is $\frac{1}{3} \cdot 2^2 \cdot 1 + \frac{1}{3} \cdot 2^2 \cdot 2 = \frac{4}{3} + \frac{8}{3} = 4$.

**Answer:** $4$
Part B

1. (a) Since $AB$ is parallel to $DC$ and $AD$ is perpendicular to both $AB$ and $DC$, then the area of trapezoid $ABCD$ is equal to $\frac{1}{2} \cdot AD \cdot (AB + DC)$ or $\frac{1}{2} \cdot 10 \cdot (7 + 17) = 120$.

   Alternatively, we could separate trapezoid $ABCD$ into rectangle $ABFD$ and right-angled triangle $\triangle BFC$.

   We note that $ABFD$ is a rectangle since it has three right angles.

   Rectangle $ABFD$ is $7$ by $10$ and so has area $70$.

   $\triangle BFC$ has $BF$ perpendicular to $FC$ and has $BF = AD = 10$.

   Also, $FC = DC - DF = DC - AB = 17 - 7 = 10$.

   Thus, the area of $\triangle BFC$ is $\frac{1}{2} \cdot FC \cdot BF = \frac{1}{2} \cdot 10 \cdot 10 = 50$.

   This means that the area of trapezoid $ABCD$ is $70 + 50 = 120$.

   (b) Since $PQ$ is parallel to $DC$, then $\angle BQP = \angle BCF$.

   We note that $ABFD$ is a rectangle since it has three right angles. This means that $BF = AD = 10$ and $DF = AB = 7$.

   In $\triangle BCF$, we have $BF = 10$ and $FC = DC - DF = 17 - 7 = 10$.

   Therefore, $\triangle BCF$ has $BF = FC$, which means that it is right-angled and isosceles. Therefore, $\angle BCF = 45^\circ$ and so $\angle BQP = 45^\circ$.

   (c) Since $PQ$ is parallel to $AB$ and $AP$ and $BT$ are perpendicular to $AB$, then $ABTP$ is a rectangle.

   Thus, $AP = BT$ and $PT = AB = 7$.

   Since $PT = 7$, then $TQ = PQ - PT = x - 7$.

   Since $\angle BQT = 45^\circ$ and $\angle BTQ = 90^\circ$, then $\triangle BTQ$ is right-angled and isosceles.

   Therefore, $BT = TQ = x - 7$.

   Finally, $AP = BT = x - 7$.

   (d) Suppose that $PQ = x$.

   In this case, trapezoid $ABQP$ has parallel sides $AB = 7$ and $PQ = x$, and height $AP = x - 7$.

   The areas of trapezoid $ABQP$ and trapezoid $PQCD$ are equal exactly when the area of trapezoid $ABQP$ is equal to half of the area of trapezoid $ABCD$.

   Thus, the areas of $ABQP$ and $PQCD$ are equal exactly when $\frac{1}{2} (x - 7)(x + 7) = \frac{1}{2} \cdot 120$,

   which gives $x^2 - 49 = 120$ or $x^2 = 169$.

   Since $x > 0$, then $PQ = x = 13$.

   Alternatively, we could note that trapezoid $PQCD$ has parallel sides $PQ = x$ and $DC = 17$, and height $PD = AD - AP = 10 - (x - 7) = 17 - x$.

   Thus, the area of trapezoid $ABQP$ and the area of trapezoid $PQCD$ are equal exactly when $\frac{1}{2} (x - 7)(x + 7) = \frac{1}{2} (17 - x)(x + 17)$, which gives $x^2 - 49 = 17^2 - x^2$ or $x^2 - 49 = 289 - x^2$

   and so $2x^2 = 338$ or $x^2 = 169$.

   Since $x > 0$, then $PQ = x = 13$. 
2. (a) The lattice points inside the region $A$ are precisely those lattice points whose coordinates $(r, s)$ satisfy $1 \leq r \leq 99$ and $1 \leq s \leq 99$.

Each point on the line with equation $y = 2x + 5$ is of the form $(a, 2a + 5)$ and so each lattice point on the line with equation $y = 2x + 5$ is of the form $(a, 2a + 5)$ for some integer $a$.

For such a lattice point to lie in region $A$, we need $1 \leq a \leq 99$ and $1 \leq 2a + 5 \leq 99$.

The second pair of inequalities is equivalent to $-4 \leq 2a \leq 94$ and thus to $-2 \leq a \leq 47$.

Since we need both $1 \leq a \leq 99$ and $-2 \leq a \leq 47$ to be true, we have $1 \leq a \leq 47$.

Since there are 47 integers $a$ in this range, then there are 47 lattice points in the region $A$ and on the line with equation $y = 2x + 5$.

These are the points $(1, 7), (2, 9), (3, 11), \ldots, (47, 99)$.

(b) Consider a lattice point $(r, s)$ that lies on the line with equation $y = \frac{5}{3}x + b$.

In this case, we must have $s = \frac{5}{3}r + b$ and so $\frac{5}{3}r = s - b$.

Since $s$ and $b$ are both integers, then $\frac{5}{3}r$ is an integer.

Since $r$ is an integer and $\frac{5}{3}r$ is an integer, then $r$ is a multiple of 3.

We write $r = 3t$ for some integer $t$ which means that $s = \frac{5}{3} \cdot 3t + b = 5t + b$.

Thus, the lattice point $(r, s)$ can be re-written as $(3t, 5t + b)$.

For $(3t, 5t + b)$ to lie within $A$, we need $1 \leq 3t \leq 99$.

Since $t$ is an integer, this means that $1 \leq t \leq 33$.

When $b = 0$, these points are the points of the form $(3t, 5t)$; these lie within $A$ when $1 \leq t \leq 19$. In other words, there are 19 points in $A$ when $b = 0$, which means that the greatest possible value of $b$ is at least 0.

We note that $5t + b$ is increasing as $t$ increases.

When $b \geq 0$ and $t \geq 1$, we have $5t + b \geq 5$ and so if any points lie within $A$, then the point with $t = 1$ must lie within $A$. This means that for at least 15 of the points $(3t, 5t + b)$ to lie within $A$, the points corresponding to $t = 1, 2, \ldots, 14, 15$ must all lie within $A$.

Since $5t + b$ is increasing, the largest value of $b$ should correspond to the largest value of $5(15) + b$ that does not exceed 99.

When $b = 24$, we note that the points for $t = 1, 2, \ldots, 14, 15$ are

$$(r, s) = (3, 29), (6, 34), \ldots, (42, 94), (45, 99)$$

which means that exactly 15 points lie within $A$.

We note that if $b \geq 25$ and $t \geq 15$, then $5t + b \geq 100$ and so the point $(3t, 5t + b)$ is not within $A$; in other words, if $b \geq 25$, there are fewer than 15 points on the line that lie within $A$.

Therefore, $b = 24$ is indeed the largest possible value of $b$ that satisfies the given requirements.
(c) Consider a line with equation \( y = mx + 1 \) for some value of \( m \).
Regardless of the value of \( m \), the point \((0, 1)\) lies on this line. This point is not in the
region \( A \), but is right next to it.
Consider the line with equation \( y = \frac{3}{7}x + 1 \) (that is, \( m = \frac{3}{7} \)).
The point \((7, 4)\) is a lattice point in \( A \) that lies on this line.
This means that \( m = \frac{3}{7} \) cannot be in the final range of values, and so \( n \) cannot be greater
than \( \frac{3}{7} \).
Consider the points on the line with equation \( y = mx + 1 \) with \( x \)-coordinates from 1 to
99, inclusive. These are the points
\[(1, m + 1), (2, 2m + 1), (3, 3m + 1), \ldots, (98, 98m + 1), (99, 99m + 1)\]
Since \( m < n \leq \frac{3}{7} \), then \( 99m + 1 < 99 \cdot \frac{3}{7} + 1 < 99 \) and so each of these 99 points are in the
region \( A \).
This means that we need to ensure that none of \( m + 1, 2m + 1, 3m + 1, \ldots, 98m + 1, 99m + 1 \)
is an integer.
In other words, we want to determine the greatest possible real number \( n \) for which none
of \( m + 1, 2m + 1, 3m + 1, \ldots, 98m + 1, 99m + 1 \) is an integer whenever \( \frac{2}{7} < m < n \).
Since real numbers \( s \) and \( s + 1 \) are either both integers or both not integers, then we want to
determine the greatest possible real number \( n \) for which none of \( m, 2m, 3m, \ldots, 98m, 99m \)
is an integer whenever \( \frac{2}{7} < m < n \).
The fact that none of \( m, 2m, 3m, \ldots, 98m, 99m \) can be an integer is equivalent to saying
that \( m \) is not equal to a rational number of the form \( \frac{c}{d} \) where \( c \) is an integer and \( d \)
is equal to one of \( 1, 2, 3, \ldots, 98, 99 \).
This means that the value of \( n \) that we want is the largest real number \( n \) with the property
that there are no rational numbers \( m = \frac{c}{d} \) with \( c \) and \( d \) integers and \( 1 \leq d \leq 99 \) in the
interval \( \frac{2}{7} < m < n \).
Let \( s \) be the smallest rational number of the form \( \frac{c}{d} \) with \( c \) and \( d \) integers and \( 1 \leq d \leq 99 \)
that is greater than \( \frac{2}{7} \).
Then it must be the case that \( n = s \).
To see why this is true, we note that \( s \) has the property that there are no rational numbers
\( m \) with the above restrictions between \( \frac{2}{7} \) and \( s \) by the definition of \( s \), and also that any
number larger than \( s \) does not have this property because \( s \) would be between it and \( \frac{2}{7} \).
Therefore, \( n = s \).
This means that we need to determine the smallest rational number of the form \( \frac{c}{d} \) with \( c \)
and \( d \) integers and \( 1 \leq d \leq 99 \) that is greater than \( \frac{2}{7} \).
To do this, we minimize the value of \( \frac{c}{d} - \frac{2}{7} = \frac{7c - 2d}{7d} \) subject to the conditions that \( c \)
and \( d \) are positive integers with \( 1 \leq d \leq 99 \) and that \( \frac{c}{d} - \frac{2}{7} = \frac{7c - 2d}{7d} > 0 \), which also
means that \( 7c - 2d > 0 \).
When \( d = 99 \), we are minimizing \( \frac{7c - 198}{693} \) which is the smallest possible when \( c = 29 \),
giving a difference of \( \frac{5}{693} \).
When \( d = 98 \), we are minimizing \( \frac{7c - 196}{686} \) which is the smallest possible when \( c = 29 \),
giving a difference of $\frac{7}{686}$.

When $d = 97$, we are minimizing $\frac{7c - 194}{679}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{2}{679}$.

When $d = 96$, we are minimizing $\frac{7c - 192}{672}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{4}{672}$.

When $d = 95$, we are minimizing $\frac{7c - 190}{665}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{6}{665}$.

When $d = 94$, we are minimizing $\frac{7c - 188}{658}$ which is the smallest possible when $c = 27$, giving a difference of $\frac{1}{658}$.

We can check that $\frac{1}{658}$ is smaller than any of $\frac{5}{693}, \frac{7}{686}, \frac{2}{679}, \frac{4}{672}, \frac{6}{665}$.

Furthermore, if $d < 94$, then since $\frac{7c - 2d}{7d} \geq \frac{1}{7d} > \frac{1}{658}$ (noting that $7c - 2d \geq 1$) and so every other difference will be greater than $\frac{1}{658}$.

This means that $\frac{27}{94}$ is the smallest of this set of rational numbers, which means that $n = \frac{27}{94}$.

3. (a) Working with $x$ in degrees

We know that $\sin \theta = 1$ exactly when $\theta = 90^\circ + 360^\circ k$ for some integer $k$.

Therefore, $\sin \left(\frac{x}{5}\right) = 1$ exactly when $\frac{x}{5} = 90^\circ + 360^\circ k_1$ for some integer $k_1$ which gives $x = 450^\circ + 1800^\circ k_1$.

Also, $\sin \left(\frac{x}{9}\right) = 1$ exactly when $\frac{x}{9} = 90^\circ + 360^\circ k_2$ for some integer $k_2$ which gives $x = 810^\circ + 3240^\circ k_2$.

Equating expressions for $x$, we obtain

$$450^\circ + 1800^\circ k_1 = 810^\circ + 3240^\circ k_2$$

$$1800k_1 - 3240k_2 = 360$$

$$5k_1 - 9k_2 = 1$$

One solution to this equation is $k_1 = 2$ and $k_2 = 1$.

These give $x = 4050^\circ$. We note that $\frac{x}{5} = 810^\circ$ and $\frac{x}{9} = 450^\circ$; both of these angles have a sine of 1.

Working with $x$ in radians

We know that $\sin \theta = 1$ exactly when $\theta = \frac{\pi}{2} + 2\pi k$ for some integer $k$.

Therefore, $\sin \left(\frac{x}{5}\right) = 1$ exactly when $\frac{x}{5} = \frac{\pi}{2} + 2\pi k_1$ for some integer $k_1$ which gives $x = \frac{5\pi}{2} + 10\pi k_1$. 
Also, \( \sin \left( \frac{x}{9} \right) = 1 \) exactly when \( \frac{x}{9} = \frac{\pi}{2} + 2\pi k_2 \) for some integer \( k_2 \) which gives \( x = \frac{9\pi}{2} + 18\pi k_2 \).

Equating expressions for \( x \), we obtain

\[
\frac{5\pi}{2} + 10\pi k_1 = \frac{9\pi}{2} + 18\pi k_2 \\
10\pi k_1 - 18\pi k_2 = 2\pi \\
5k_1 - 9k_2 = 1
\]

One solution to this equation is \( k_1 = 2 \) and \( k_2 = 1 \).

These give \( x = \frac{45\pi}{2} \). We note that \( \frac{x}{5} = \frac{9\pi}{2} \) and \( \frac{x}{9} = \frac{5\pi}{2} \); both of these angles have a sine of 1.

Therefore, one solution is \( x = 4050^\circ \) (in degrees) or \( x = \frac{45\pi}{2} \) (in radians).

(b) Suppose that \( M \) and \( N \) are positive integers.

We work towards determining conditions on \( M \) and \( N \) for which there is or is not an angle \( x \) with \( \sin \left( \frac{x}{M} \right) + \sin \left( \frac{x}{N} \right) = 2 \).

Since \(-1 \leq \sin \theta \leq 1\) for all angles \( \theta \), then the equation \( \sin \left( \frac{x}{M} \right) + \sin \left( \frac{x}{N} \right) = 2 \) is equivalent to the pair of equations \( \sin \left( \frac{x}{M} \right) = \sin \left( \frac{x}{N} \right) = 1 \). (Putting this another way, there must be an angle \( x \) which makes both sines 1 simultaneously.)

As in (a), the equation \( \sin \left( \frac{x}{M} \right) = 1 \) is equivalent to the statement that \( \frac{x}{M} = 90^\circ + 360^\circ r \) or \( \frac{x}{M} = \frac{\pi}{2} + 2\pi r \) for some integer \( r \). (We will carry equations in degrees and in radians simultaneously for a time.)

These equations are equivalent to saying \( x = 90^\circ M + 360^\circ r M \) or \( x = \frac{M\pi}{2} + 2\pi r M \) for some integer \( r \).

Similarly, the equation \( \sin \left( \frac{x}{N} \right) = 1 \) is equivalent to saying \( x = 90^\circ N + 360^\circ s N \) or \( x = \frac{N\pi}{2} + 2\pi s N \) for some integer \( s \).

Since \( x \) is common, then we can equate values of \( x \) to say that if such an \( x \) exists, then \( 90^\circ M + 360^\circ r M = 90^\circ N + 360^\circ s N \) or \( \frac{M\pi}{2} + 2\pi r M = \frac{N\pi}{2} + 2\pi s N \).

It is also true that if these equations are true, then the existence of an angle \( x \) that satisfies, say, \( x = 90^\circ M + 360^\circ r M \) then guarantees the fact that the same angle \( x \) satisfies \( x = 90^\circ N + 360^\circ s N \).

In other words, the existence of an angle \( x \) is equivalent to the existence of integers \( r \) and \( s \) for which \( 90^\circ M + 360^\circ r M = 90^\circ N + 360^\circ s N \) or \( \frac{M\pi}{2} + 2\pi r M = \frac{N\pi}{2} + 2\pi s N \).

Dividing the first equation throughout by \( 90^\circ \) and the second equation throughout by \( \frac{\pi}{2} \) gives us the same resulting equation, namely \( M + 4r M = N + 4s N \). Thus, we can not concern ourselves with using degrees or radians for the rest of this part.

At this stage, we know that there is an angle \( x \) with the desired property precisely when there are integers \( r \) and \( s \) for which \( M + 4r M = N + 4s N \).
Suppose that $M = 2^a c$ and $N = 2^b d$ for some integers $a$, $b$, $c$, $d$ with $a \geq 0$, $b \geq 0$, $c$ odd, and $d$ odd. Here, we are writing $M$ and $N$ as the product of a power of 2 and their “odd part”.

Suppose that $a \neq b$; without loss of generality, assume that $a > b$. Then, the following equations are equivalent:

\[
M + 4rM = N + 4sN \\
2^a c + 4r \cdot 2^a c = 2^b d + 4s \cdot 2^b d \\
2^{a-b} c + 2^{2+a-b} rc = d + 4sd \\
2^{a-b} c + 2^{2+a-b} rc - 4sd = d
\]

Since the right side of this equation is an odd integer and the left side is an even integer regardless of the choice of $r$ and $s$, there are no integers $r$ and $s$ for which this is true.

Thus, if $M$ and $N$ do not contain the same number of factors of 2, there is no angle $x$ that satisfies the initial equation.

To see this in another way, we return to the equation $M + 4rM = N + 4sN$, factor both sides to obtain $M(1 + 4r) = N(1 + 4s)$ which gives the equivalent equation $\frac{M}{N} = \frac{1 + 4s}{1 + 4r}$.

If integers $r$ and $s$ exist that satisfy this equation, then $\frac{M}{N}$ can be written as a ratio of odd integers and so $M$ and $N$ must contain the same number of factors of 2.

Putting this another way, if $M$ and $N$ do not contain the same number of factors of 2, then integers $r$ and $s$ do not exist and so the initial equation has no solutions.

To complete (b), we need to demonstrate the existence of a sequence $n_1, n_2, \ldots, n_{100}$ of positive integers for which $\sin \left( \frac{x}{n_i} \right) + \sin \left( \frac{x}{n_j} \right) \neq 2$ for all angles $x$ and for all pairs $1 \leq i < j \leq 100$.

Suppose that $n_i = 2^i$ for $1 \leq i \leq 100$.

In other words, the sequence $n_1, n_2, \ldots, n_{100}$ is the sequence $2^1, 2^2, \ldots, 2^{100}$.

No pair of numbers from the sequence $n_1, n_2, \ldots, n_{100}$ contains the same number of factors of 2, and so there is no angle $x$ that makes $\sin \left( \frac{x}{n_i} \right) + \sin \left( \frac{x}{n_j} \right) = 2$ for any $i$ and $j$ with $1 \leq i < j \leq 100$.

Therefore, the sequence $n_i = 2^i$ for $1 \leq i \leq 100$ has the desired property.

(c) Suppose that $M$ and $N$ are positive integers for which there is an angle $x$ that satisfies the equation $\sin \left( \frac{x}{M} \right) + \sin \left( \frac{x}{N} \right) = 2$.

From (b), we know that $M$ and $N$ must contain the same number of factors of 2.

Again, suppose that $M = 2^a c$ and $N = 2^a d$ for some integers $a$, $c$, $d$ with $a \geq 0$, $c$ odd, and $d$ odd.

Then, continuing from earlier work, the following equations are equivalent:

\[
M + 4rM = N + 4sN \\
2^a c + 4r \cdot 2^a c = 2^a d + 4s \cdot 2^a d \\
c + 4rc = d + 4sd \\
c - d = -4rc + 4sd
\]

Since the right side is a multiple of 4, then the left side must also be a multiple of 4 and so $c$ and $d$ have the same remainder when divided by 4.
(Using a more advanced result from number theory, it turns out that if \( c - d \) is divisible by 4, then this equation will always have a solution for the integers \( r \) and \( s \), but we do not need this precise fact.)

Suppose that \( m_1, m_2, \ldots, m_{100} \) is a list of 100 distinct positive integers with the property that, for each integer \( i = 1, 2, \ldots, 99 \), there is an angle \( x_i \) that satisfies the equation

\[
\sin \left( \frac{x_i}{m_i} \right) + \sin \left( \frac{x_i}{m_{i+1}} \right) = 2.
\]

Suppose further that \( m_1 = 6 \).

Since \( m_1 = 2^1 \cdot 3 \) and there is an angle \( x_1 \) with \( \sin \left( \frac{x_1}{m_1} \right) + \sin \left( \frac{x_1}{m_2} \right) = 2 \), then from above, \( m_2 = 2^1 \cdot c_2 \) for some positive integer \( c_2 \) that is 3 more than a multiple of 4 (that is, \( c_2 \) has the same remainder upon division by 4 as 3 does).

Similarly, each integer in the list \( m_1, m_2, \ldots, m_{100} \) can be written as \( m_i = 2c_i \) where \( c_i \) is a positive integer that is 3 more than a multiple of 4.

Define \( t = \frac{3\pi}{2^{100}} \cdot m_1 m_2 \cdots m_{100}. \)

Then

\[
\frac{t}{m_i} = \frac{3\pi}{2 \cdot 2^{99} (2c_i)} (2c_1)(2c_2) \cdots (2c_{100}) = \frac{\pi}{2} \cdot \frac{3c_1 c_2 \cdots c_{100}}{c_i}.
\]

In other words, \( \frac{t}{m_i} \) is equal to \( \frac{\pi}{2} \) times the product of 100 integers each of which is 3 more than a multiple of 4. (Note that the numerator of the last fraction includes 101 such integers and the denominator includes 1.)

The product of two integers each of which is 3 more than a multiple of 4 is equal to an integer that is 1 more than a multiple of 4. This is because if \( y \) and \( z \) are integers, then

\[
(4y + 3)(4z + 3) = 16yz + 12y + 12z + 9 = 4(4yz + 3y + 3z + 2) + 1
\]

Also, the product of two integers each of which is 1 more than a multiple of 4 is equal to an integer that is 1 more than a multiple of 4. This is because if \( y \) and \( z \) are integers, then

\[
(4y + 1)(4z + 1) = 16yz + 4y + 4z + 1 = 4(4yz + y + z) + 1
\]

Thus, the product of 100 integers each of which is 3 more than a multiple of 4 is equal to the product of 50 integers each of which is 1 more than a multiple of 4, which is equal to an integer that is one more than a multiple of 4.

Therefore, \( \frac{t}{m_i} \) is equal to \( \frac{\pi}{2} \) times an integer that is 1 more than a multiple of 4, and so

\[
\sin \left( \frac{t}{m_1} \right) = 1,
\]

and so

\[
\sin \left( \frac{t}{m_1} \right) + \sin \left( \frac{t}{m_2} \right) + \cdots + \sin \left( \frac{t}{m_{100}} \right) = 100
\]

as required.

Therefore, for every such sequence \( m_1, m_2, \ldots, m_{100} \), there does exist an angle \( t \) with the required property.