WATERLOO MATHEMATICS
University of Waterloo
Faculty of Mathematics

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Let's Solve Some Problems!

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What is a "Problem"?

- Less "direct" than an exercise
- Often combines multiple ideas or multiple areas
- Not necessarily a word problem

Prerequisites for Problem Solving

- Number sense and number skills
- Pattern recognition
- Patience, humility, ability to learn from mistakes, perseverance

Preliminary Steps

- Read the problem carefully and understand each word in the statement
- Understand what is given and what is asked for
- Search memory for relevant and important facts
- Search memory for similar problems that you have previously solved

Solving the Problem

Final Steps

- Did I get an answer?
- Does my answer actually answer the question?
- Does my answer make sense?
- Write up a clear solution, including sufficient detail for the audience

How to Become a Better Problem Solver

- Practice, practice, practice
- Work with better problem solvers
- Try problems before reading solutions
- Remember your mistakes
- Make up your own problems

Trial and Error

- Sometimes a good place to start, rarely a good place to end
- More useful in multiple choice than full solution
- Works when asked to find an answer, but rarely when asked to find all answers
- Not helpful when answers are not small integers
- Requires thinking and adjusting

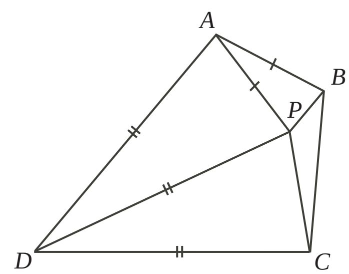
Three pumpkins are weighed two at a time in all possible ways. The masses of the pairs of pumpkins are 12 kg, 13 kg and 15 kg. What is the mass of the lightest pumpkin?

Determine a pair of positive integers (x, y) with $x \leq 5$ for which $x^2 - 2y^2 = 1$.

Just do it

- Sometimes, there are no really surprising steps and the plan of attack is not too tricky.
- The definition of "no really surprising" steps will of course vary from person to person.
- Here, use logic and facts that you know to proceed more or less in a straight line.
- With practice, we develop a sixth sense that allows us to intuit that there will be an answer at the end, even if we don't quite see how we will get there.

In the diagram, point P is inside quadrilateral $ABCD$. Also, $DA = DP = DC$ and $AP = AB$. If $\angle ADP = \angle CDP = 2x^\circ$, $\angle BAP = (x + 5)^\circ$, and $\angle BPC = (10x - 5)^\circ$, what is the value of x?



In the addition of the three-digit numbers shown, the letters A, B, C, D, and E each represent a single digit.

$$\begin{array}{cccc} & A & B & E \\ & A & C & E \\ + & A & D & E \\ \hline 2 & 0 & 1 & 7 \end{array}$$

What is the value of $A + B + C + D + E$?

Draw a useful diagram

- A diagram might mean visual representation
- Size matters
- Put appropriate labels on it
- Easy to draw, but not too detailed

The eight vertices of a cube are labelled with the integers from 1 to 8 inclusive. Judith looks at the labels of the four vertices of one of the faces of the cube. She lists these four labels in increasing order. After doing this for all six faces, she gets the following six lists: $(1, 2, 5, 8)$, $(3, 4, 6, 7)$, $(2, 4, 5, 7)$, $(1, 3, 6, 8)$, $(2, 3, 7, 8)$, and $(1, 4, 5, 6)$. What is the label of the vertex of the cube that is farthest away from the vertex labelled 2?

Car X and Car Y are travelling in the same direction in two different lanes on a long straight highway. Car X is travelling at a constant speed of 90 km/h and has a length of 5 m. Car Y is travelling at a constant speed of 91 km/h and has a length of 6 m. Car Y starts behind Car X and eventually passes Car X. The length of time between the instant when the front of Car Y is lined up with the back of Car X and the instant when the back of Car Y is lined up with the front of Car X is t seconds. What is the value of t?

Pascal High School organized three different trips. Fifty percent of the students went on the first trip, 80% went on the second trip, and 90% went on the third trip. A total of 160 students went on all three trips, and all of the other students went on exactly two trips. How many students are at Pascal High School?

Use Algebra

- Some problems have explicit algebraic components
- Other problems do not, but are well-suited to introducing algebra
- How do we know?
  - Relationships between various quantities given
  - Quantities that we want to find appear to be less directly related to the information given
- Good approach to consider if you are someone who defaults to Trial and Error
- There are dangers
  - Introducing too many variables and so over-complicating the problem
  - Missing a non-algebraic solution

Exactly 120 tickets were sold for a concert. The tickets cost $12 each for adults, $10 each for seniors, and $6 each for children. The number of adult tickets sold was equal to the number of child tickets sold. The total revenue from the ticket sales was $1100. What was the number of senior tickets sold?

In the table, the numbers in each row form an arithmetic sequence when read from left to right. Similarly, the numbers in each column form an arithmetic sequence when read from top to bottom. What is the value of x?

				18
	43			
		40		
x			26	

Finding a pattern

Finding a pattern can help us to

- Solve a problem
- Find the next step
- Make a guess as to what is happening

What are the next three terms in the sequence $1, 2, 3, 4, \ldots$?

The expression $n^2 - n + 41$ equals a prime number for each integer n with $1 \leq n \leq 40$.
Is $n^2 - n + 41$ a prime number for every positive integer?

What is the units digit of 3^{2018}?

In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

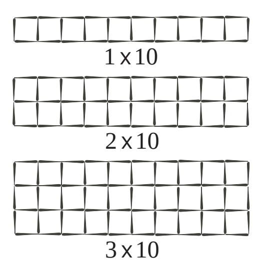
Daryl first writes the perfect squares as a sequence

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

After the number 1, he then alternates by making two terms negative followed by leaving two terms positive. Daryl's new sequence is

$$1, -4, -9, 16, 25, -36, -49, 64, 81, -100, \ldots$$

What is the sum of the first 2018 terms in this new sequence?

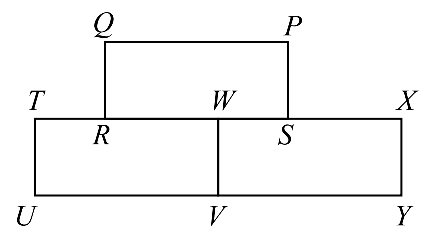


Toothpicks are used to make rectangular grids, as shown. Note that a total of 31 identical toothpicks are used in the 1×10 grid. How many toothpicks are used in a 43×10 grid?

Try a simpler problem

- More likely to help find an answer rather than a solution
- Can be useful in multiple choice problems
- Might give us some insight as to what is happening in more general problem
- One way in which we often "Try a simpler problem" is in looking at small cases and then finding a pattern.

An integer x is chosen so that $3x + 1$ is an even integer. Which of the following must be an odd integer?
(A) $x + 3$ (B) $x - 3$ (C) $2x$ (D) $7x + 4$ (E) $5x + 3$



Three identical rectangles $PQRS$, $WTUV$ and $XWVY$ are arranged, as shown, so that RS lies along TX. The perimeter of each of the three rectangles is 21 cm. What is the perimeter of the whole shape?

Working systematically

- Always work systematically
- Work through the given information systematically.
- Work through possibilities systematically.

Each of the positive integers 2013 and 3210 has the following three properties:

(i) it is an integer between 1000 and 10 000,

(ii) its four digits are consecutive integers, and

(iii) it is divisible by 3.

In total, how many positive integers have these three properties?

A 51 cm rod is built from 5 cm rods and 2 cm rods. All of the 5 cm rods must come first, and are followed by the 2 cm rods. For example, the rod could be made from seven 5 cm rods followed by eight 2 cm rods. How many ways are there to build the 51 cm rod?

In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that $x < y < z$ are constructed. How many such triangles are there?

Working Backwards

Working backwards can mean trying to "reverse engineer" a scenario by starting with this scenario and trying to:

- logically deduce the steps that led to it, or
- making an education guess as to what steps could have led to it.

Angie has a jar that contains 2 red marbles, 2 blue marbles, and no other marbles. She randomly draws 2 marbles from the jar. If the marbles are the same colour, she discards one and puts the other back into the jar. If the marbles are different colours, she discards the red marble and puts the blue marble back into the jar. She repeats this process a total of three times. What is the probability that the remaining marble is red?

In the country of Nohills, each pair of cities is connected by a straight (and flat) road. The chart to the right shows the distances along the straight roads between some pairs of cities. What is the distance along the straight road between city P and city R?

	P	Q	R	S
P	0	25		24
Q	25	0	25	7
R		25	0	18
S	24	7	18	0

Problem Solving Extras

1. For how many triples (B, C, D) of non-negative integers is $B + C + D = 9$?

2. For how many triples (B, C, D) of non-negative digits is $B + C + D = 19$?

3. How many toothpicks are used to make a $4 \times 4 \times 4$ three-dimensional lattice? (Imagine using organic marshmallows to hold each vertex together.)

4. How many ways are there of making \$1.00 using quarters (worth \$0.25 each), dimes (worth \$0.10 each), and nickels (worth \$0.05 each)? (Any given combination does not necessarily need to include each type of coin.)

5. In the table, the result from this year's Waterloo Winter Walleye Weekend fishing derby are shown. We can see how many contestants caught f fish for various values of f:

f	0	1	2	3	$\cdots$	13	14	15
number of contestants who caught f fish	9	5	7	23	$\cdots$	5	2	1

In the Imprint story about the event, it was (correctly) reported that

- the winner caught 15 fish,
- the fisherpeople who caught 3 or more fish averaged 6 fish each, and
- the fisherpeople who caught 12 or fewer fish averaged 5 fish each.

Determine the total number of fish caught during the WWWW.

6. Determine the number of even four-digit positive integers between 2000 and 5000 that consist of four distinct digits.

7. Zach and Yun are playing a game, starting with a pack of 60 cards. Zach begins by discarding at least one but not more than half of the cards in the pack. He then passes the remaining cards in the pack to Yun. Yun continues the game by discarding at least one but not more than half of the remaining cards in the pack. The game continues in this way with the pack being passed back and forth between the two players. The loser is the player who, at the beginning of his or here turn, receives only one card. Determine which player has a winning strategy and describe this strategy.

8. Five teams played in a competition and every team played against each of the other four teams exactly once. Each team received 3 points for a win, 1 point for a tie, and no points for a loss. At the end of the competition the points were:

Yellow 10, Red 9, Green 4, Blue 3, Pink 1

Determine how many matches ended in a tie.

9. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

10. In a sequence of positive integers, every term after the first two terms is the sum of the two previous terms in the sequence. If the sixth term is 2011, then determine the maximum possible value of the first term.