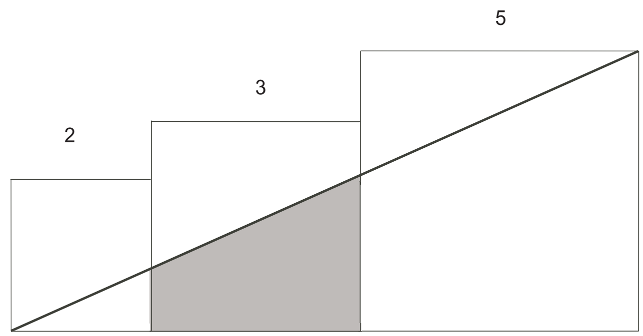
University of Waterloo
Faculty of Mathematics

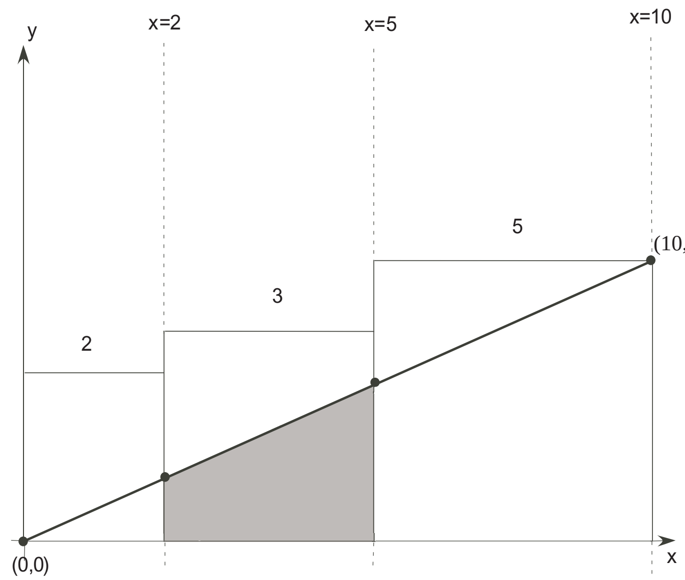
Centre for Education in
Mathematics and Computing

# Intermediate Math Circles
# April 1, 2009
# Analytic Geometry II

Exercise: Three squares, of side length $2, 3$ and $5$ respectively, are placed side by side as shown. What is the area of the shaded quadrilateral?




Solution:
Similar triangles can be used to solve this problem but analytic geometry is a clean and straightforward alternative approach.
We will use the cartesian plane to attach coordinates to the diagram.
Call the bottom left corner of our diagram the origin. Thus we know that our line passes through the origin, the point $(10, 5)$, and the vertical lines $x = 2$, and $x = 5$.
Since we have the coordinates of two points on the line we can find the slope of the line to get the equation for the line.
$m = \frac{5-0}{10-0} = \frac{1}{2}$. Thus the equation for the line is $y = \frac{1}{2}x$.
Therefore we know the coordinate of the other two points on the line: $(2, 1)$ and $(5, \frac{5}{2})$.




The area of the shaded region is equal to the area of the triangle enclosed by the lines $y = \frac{1}{2}x, x = 5$, and the $x$-axis, minus the area of the triangle enclosed by $y = \frac{1}{2}x, x = 2$ and the $x$-axis.
$A(shaded) = \frac{1}{2}\frac{5}{2}(5) - \frac{1}{2}(1)(2) = \frac{25}{4} - 1 = \frac{21}{4}$

**Intersection of Lines**
It is very useful for us to find out where two lines meet. This is called their intersection point. For example, an air traffic controller would want to know if the paths of two planes would ever meet. This way collisions can be avoided.
It is easy for us to find when one of the lines is horizontal or vertical, but what about lines of the form $y = mx + b$?

This is an algebraic problem in two unknowns. We have two lines and two unknown variables, $x$ and $y$. So we can solve the system of equations.
Let $(x_o, y_o)$ be the point of intersection. We know that this point must lie on both lines.
Therefore,
$y_o = ax_o + b$
$y_o = cx_o + d$
Therefore, we have that $ax_o + b = cx_o + d$.
Thus to solve a system of two equations in two unknowns we just set the two equations equal to one another and solve for $x$. Once we have $x$ we can substitute back in to find $y$.
You can easily check your answer by substituting the point back into both equations and making sure the point satisfies both equations.

Note: You cannot solve a system of two equations in three unknowns.

Example: Find the point of intersection between the lines $y = \frac{2}{3}x + 2$ and $y = -\frac{1}{3}x + 5$.

Solution:
Since we have two equations and two unknowns ($x$ and $y$) we can solve this system.
Set the equations equal to one another.
$\frac{2}{3}x + 2 = -\frac{1}{3}x + 5$
Solving we get
$x = 3$
Substituting back into one of our equations we get $y = \frac{2}{3}(3) + 2 = 2 + 2 = 4$.
We can substitute into the other equation to check our answer. $y = -\frac{1}{3}3 + 5 = -1 + 5 = 4$.
Therefore the point of intersection is $(3, 4)$.

Using this approach you can always find the point of intersection of two lines. If one exists.

**Equations of Lines**

We have already seen the slope/intercept equation for a line. However, there are many more ways in which we can define or describe lines.

Point-Slope Form of a line: $\mathbf{(y - y_o) = m(x - x_o)}$
Instead of using the $y$-intercept as our point, we use any point on the line.
For example, given the slope/intercept equation $y = 3x + 5$, we know that the point $(-1, 2)$ lies on the line. Therefore we can write it in point-slope form as $y - 2 = 3(x + 1)$.
Be careful of the signs!!! They are always opposite.

Two Point Version: $\mathbf{y - y_o = \frac{y_2 - y_1}{x_2 - x_1}(x - x_o)}$
This is the same as the point-slope form, except we are figuring out the slope beforehand.

Intercept Form: $\mathbf{\frac{x}{a} + \frac{y}{b} = 1}$. Where $a$ is the $x$-intercept and $b$ is the $y$-intercept.
For example, $y = x + 3$ could be written in Intercept form as $\frac{x}{-3} + \frac{y}{3} = 1$.

Standard Form: $\mathbf{Ax + By + C = 0}$
This is a rearrangement of the slope-intercept from for the equation of a line.

For example, $y = \frac{3}{4}x + 5$ would be written in standard form as $-\frac{3}{4}x + y - 5 = 0$.

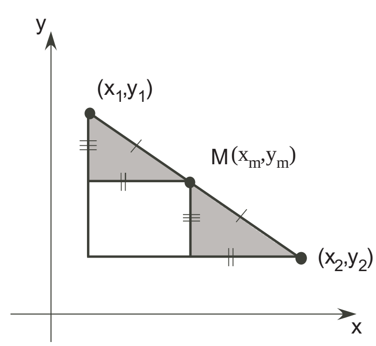
Exercise: Find the slope, $y$-intercept, and $x$-intercept of a line in Standard form.

Solution:
We know the slope and $y$-intercept for a line of the form $y = mx + b$, therefore we need to convert the line from Standard form back to the slope-intercept form.
$By = -Ax - C$
$y = -\frac{A}{B}x - \frac{C}{B}$.
Therefore the slope of a line in Standard form is $-\frac{A}{B}$, and the $y$-intercept is $-\frac{C}{B}$.
To find the $x$-intercept we set $y = 0$. Giving us $0 = -\frac{A}{B}x - \frac{C}{B}$.
$\frac{C}{B} = -\frac{A}{B}x$
$x = -\frac{C}{A}$.
Therefore a line in Standard form has $x$-intercept is $-\frac{C}{A}$, $y$-intercept $-\frac{C}{B}$, and slope $-\frac{A}{B}$.

**Midpoint**
We have already seen how to find the length of a line segment. Another useful tool, when dealing with problems in analytic geometry is, the midpoint. We are interested in finding the point on the line segment that is equidistant from both endpoints.

If $(x_1, y_1)$ and $(x_2, y_2)$ are the two endpoints of the line segment, then the midpoint is:
$(\mathbf{\frac{x_1+x_2}{2}}, \mathbf{\frac{y_1+y_2}{2}})$
Intuitively this makes sense. We are taking the average of the $x$ and $y$ coordinates.

Here is a sketch proof:
If $M(x_m, y_m)$ is the midpoint of the line with endpoints $(x_1, y_1)$ and $(x_2, y_2)$, as shown, we can use parallel lines to show that the two shaded triangles are congruent.
$x_m = x_2 - (x_2 - x_m)$ but since we have congruent triangles:
$x_m = x_2 - (x_m - x_1)$




Solving we get that $x_m = \frac{x_1+x_2}{2}$
Using a similar argument we get that $y_m = \frac{y_1+y_2}{2}$ $\square$

Example: Find the midpoint of the line segment with endpoints $(2, 6)$ and $(12, 2)$.

Solution:
Substituting the endpoints into the formula for the midpoint we get:
$(\frac{12+2}{2}, \frac{2+6}{2})$
$= (7, 4)$
Therefore the midpoint of the line segment is the point $(7, 4)$.

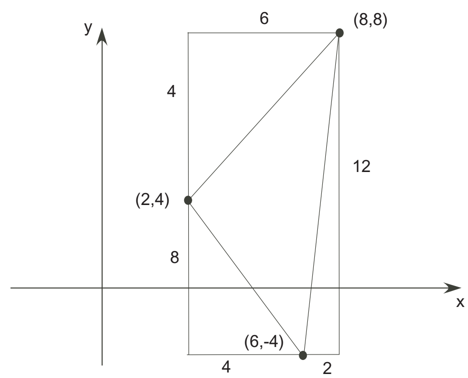
We can now use our distance formula to check our answer.
If $(7, 4)$ really is the midpoint, then the distance from one of the endpoints to the midpoint must be equal to the distance from the other endpoint to the midpoint. We have already seen how to find

the distance between points.
recall: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
therefore we have:
$d = \sqrt{(7-2)^2 + (4-6)^2} = \sqrt{29}$
and
$d = \sqrt{(7-12)^2 + (4-2)^2} = \sqrt{29}$
Hence, we have verified that $(7, 4)$ is the midpoint of the line segment.

**Triangles**
Given three points in the cartesian plane (not all in a straight line), these points can form a triangle. It is again useful for us to know the area of the triangle formed by any three points.

Example: Find the area of the triangle formed by the three points $(8, 8)$, $(2, 4)$ and $(6, -4)$.

Solution:
First we plot our points and draw a rectangle around the triangle. We can now break our problem into simpler problems.
The area of the triangle is equal to the total area of the rectangle minus the area of the three right angled triangle. Because of our coordinate system we know the side lengths of the rectangle to be 12 and 6. Similarly, we can figure our the side lengths of the right angled triangles.




Now we have:
$A = (12)(6) - \frac{1}{2}6(4) - \frac{1}{2}12(2) - \frac{1}{2}8(4) = 72 - 12 - 12 - 16 = 32.$

In General, if the points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are the three vertices of a triangle, then:
$\mathbf{A = \frac{1}{2}|x_1x_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|}$

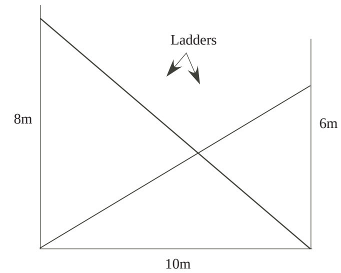
Substitute the coordinates into the formula.
$A = \frac{1}{2}|2(8) + 8(-4) + 6(4) - 8(4) - 6(8) - 2(-4) = \frac{1}{2}|-64| = \frac{1}{2}(64) = 32$ as expected.

Another way with which to solve this problem is to translate the coordinates so that one of the coordinates is the origin. This way four of the terms in our formula will disappear and we will only be left with two.

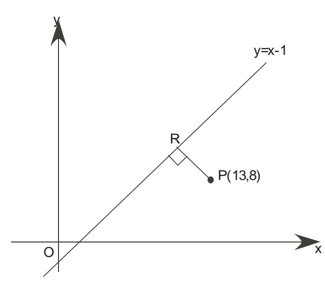
In our example, translate the coordinate $(2, 4)$ to the origin. After performing this translation we have the three coordinates $(0, 0)$, $(6, 4)$, and $(4, -8)$. Substituting these into our formula we get:
$A = \frac{1}{2}|6(-8) - 4(4)| = \frac{1}{2}(64) = 32$ as expected.

**Problems**

1. Calculate the area of the triangle with vertices $A(1,3), B(7,11)$, and $C(9,8)$.
2. Calculate the area of $\triangle ABC$ with vertices $A(-3,1), B(5,1)$ and $C(8,7)$.
3. In an alleyway 10 metres wide, two ladders are placed as shown. The first reaches a height of 6 metres above the pavement and the second a height of 8 metres. At what height do the ladders cross?




4. Determine the values of $t$ and $k$ so that the lines represented by $tx-2y+k=0$ and $3x+y-t=0$ intersect at $(-2,1)$.
5. Points $P$, $Q$, and $R$ divide the line segment from $A(0,2)$ to $C(6,0)$ (in that order) into four equal parts. Find the slope of $OR$, where $O$ is the origin.
6. A straight line $l$ passes through the midpoint, $M$, of the line from $A(-3,4)$ to $B(7,-2)$. Find the equation of the straight line that passes through $M$ and has a slope that is one more than the slope of $AB$.
7. A triangle $T$ is formed by the vertices $(0,0), (a,0)$, and $(b,c)$. A second triangle $U$ is formed by the three midpoints of the sides of $T$. Show that the area of $U$ is one fourth the area of $T$.
8. A point $P$ is chosen on the line $y=2x+3$ and a point $Q$ is chosen on $y=-x+2$. If the midpoint $M$ of the line segment $PQ$ is $(2,5)$ calculate the coordinates of $P$ and $Q$.
9. Determine whether the right bisector of the line segment from $A(2,5)$ to $B(8,1)$ passes through the point $T(-8,-17)$.
10. Find the equation of the straight line containing the point $(-3,1)$, and parallel to the line passing through the two points $(0,-2), (5,2)$.
11. In the diagram $PR$ is perpendicular to the line $y=x-1$. Determine the coordinates of the point $R$.




12. The vertices of a quadrilateral are $(0,0), (2,4), (6,7), (8,0)$. Find the equations of its sides.
13. The points $(5,3)$ and $(1,-1)$ are plotted on a sheet of graph paper. The sheet of graph paper is folded along a line so that the point $(5,3)$ lands on top of the point $(1,-1)$. The equation of the line that represents the fold is.
14. For each value of $x$, $f(x)$ is defined to be the minimum value of the three numbers $2x+2$, $\frac{1}{2}x+1$ and $-\frac{3}{4}x+7$. Find the maximum value of $f(x)$.