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Intermediate Math Circles
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Sequences and Series

Tower of Hanoi

The Tower of Hanoi is a game played with three pegs and discs of increasing size. At the start of the game, all of the discs are one one peg and arranged by size, with the smallest on top. The object of the game is to stack the discs on a different peg in the same order as you started with in the fewest number of moves. The rules for moving discs are:

- You may only move one disc at a time.
- You may move only a disc that is alone on a peg, or one that is at the top of a pile.
- You may place a disc only on an open peg, or on top of another disc that is larger than it.

Questions:

1. Determine the minimum number of moves required for 1, 2 and 3 discs.
2. Verify your result with a neighbour.
3. In general, if there are n discs, what is the minimum number of moves required to complete the game?

Recursive Sequences:

A **recursive sequence** is a sequence where one or more terms is given and the next term is determined by the previous terms.

If a_n is the minimum number of moves required to move n discs, the solution to the Tower of Hanoi problem is:

$a_1 = 1$, and $a_n = 2a_{n-1} + 1$

Exercise: The sequence of numbers $a_1, a_2, a_3, ...$ is defined by

$a_1 = 2$ and

$a_n = \dfrac{a_{n-1} - 1}{a_{n-1} + 1}$, if n is an integer greater than 1.

1. Determine a_2, a_3, a_4, a_5, and a_6
2. Determine the value of a_{999}

Another well known recursive sequence is the Fibonacci sequence:

$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, ...$

Can you see the pattern here?

Summary:

A **sequence** is an ordered list of numbers.

A sequence may be arithmetic, geometric, or neither.

How do you know if a sequence is arithmetic?

Check to see if consecutive terms differ by the same constant.

The nth term of an **arithmetic** sequence with first term a and common difference d is $a_n = a+(n-1)d$

How do you know if a sequence is geometric?

Check to see if the ratio between consecutive terms is a constant.

The nth term of a **geometric** sequence with first term a and common ratio r is
$a_n = ar^{n-1}$

Exercise: The sequence $6, -2, x, y$ is such that the first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Find the values of x and y.

A **series** is the sum of terms in a sequence.

We denote the sum of the first n terms by $S_n = a_1 + a_2 + a_3 + ... + a_n$

For an **arithmetic series**, to determine S_n, the sum of the first n terms, either:

1. Write out the series twice, once forward and once backward, one above the other. Add pairs of terms together.
2. Use one of the two formulas: $S_n = \dfrac{n(2a + (n-1)d)}{2}$ or $S_n = \dfrac{n(a_1 + a_n)}{2}$

One arithmetic sequence that comes up often is $1 + 2 + 3 + 4 + ... + n = \dfrac{n(n+1)}{2}$

For a **geometric series**, to determine S_n, the sum of the first n terms:

Use one of the two formulas: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ or $S_n = \dfrac{(a_1 - a_{n+1})}{1-r}$

Bricks are stacked in 16 rows such that each row has a fixed number of bricks less than the row below it. The top row has 7 bricks and the bottom row has 52 bricks. How many bricks are in the stack?

The first triangular number is $t_1 = 1$, the second is $t_2 = 3$, the third is $t_3 = 6$, the fourth is $t_4 = 10$, and the nth triangular number is $t_n = 1 + 2 + 3 + ... + (n-1) + n$. Calculate the 7th and 20th triangular numbers.

The sum of 25 consecutive *odd* integers is 1175. What is the smallest of these integers?

What?!? 0.99999... = 1??

More Problems:

1. In a sequence of positive numbers, each term after the first two is the sum of *all* the previous terms. If the first term is a, the second term is 2, and the sixth term is 56, then determine the value of a.

2. One plant is now 12 cm tall and will grow 2 cm per week. A second plant is now 3 cm tall and will grow 5 cm per week. How many weeks does it take before the plants are the same height?

3. Determine the smallest positive integer x for which the sum $x + 2x + 3x + 4x + ... + 100x$ is a perfect square.

4. The multiples of 2 and 5 are removed from the set of positive integers $1, 2, 3, ..., 10n$, where n is an integer. Determine the sum of the remaining integers.

5. In a sequence of six numbers, the first number is 4 and the last number is 47. Each number after the second equals the sum of the *two* previous numbers. If S is the sum of the six numbers in this sequence, then what is the value of S?

6. Consider the sequence $a_1, a_2, a_3, ...$ with general term a_n. Given that the sum of the first n terms of the sequence is equal to $5n^2 + 6n$,

   a) determine S_1, S_2, and S_3.

   b) determine a_1, a_2, and a_3.

   c) determine a_n.

7. The sum of the first n terms of a sequence is $n(n+1)(n+2)$. Determine the 10th term of this sequence. Determine the 101th.

8. Consider the sequence $t_1 = 1, t_2 = -1$, and $t_n = \left(\frac{n-3}{n-1}\right) t_{n-2}$, where $n \geq 3$. What are the values of t_{2008} and t_{2009}?

9. If $a_{n+1} = \frac{2a_n + 1}{2}$ and $a_1 = 1$, find a_{235}.