University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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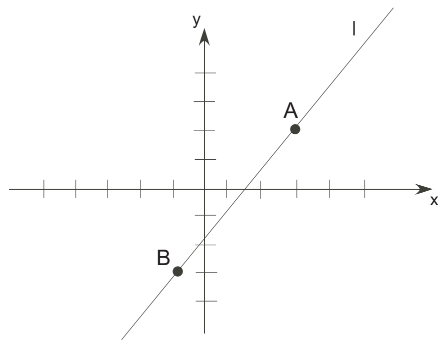
Analytic Geometry I

Analytic Geometry is the use of a coordinate system to translate a geometry problem into an algebraic problem. Analytic Geometry is NOT using angle theorems, similar triangles, and trigonometry. Instead we will be using points and equations for lines.

To solve problems using analytic geometry we have to understand the basics:
The coordinate system we will be using is the Cartesian coordinate system, the xy-plane.
Our problems will consist of using the cartesian plane along with points, lines and line segments.



From the diagram, A and B are the points corresponding to the coordinates $(3, 2)$ and $(-1, -3)$ respectively.

l is the line passing through points A and B.

AB is the line segment with endpoints A and B.
The difference between a line and a line segment is that a line segment has a fixed length.

These are the tools we will be using to solve problems in analytic geometry.

Distance Between Points
For many problems we need to know the distance between two points, the length of a line segment.

If d is the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ then:
$\mathbf{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

Example: Find the distance between points A and B above.

Solution:
We have the points $A(3, 2)$ and $B(-1, -3)$. Substituting into our formula gives us,

$d = \sqrt{(-1-3)^2 + (-3-2)^2} = \sqrt{16+25} = \sqrt{41}$
Therefore line segment AB has length $\sqrt{41}$.

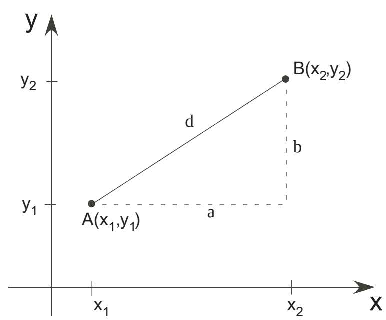
But what happens if we choose point A to be (x_2, y_2)?

$d = \sqrt{(3 - -1)^2 + (2 - -3)^2} = \sqrt{16+25} = \sqrt{41}$

Therefore we can see that it does not matter which point we choose to be (x_1, y_1) and which one is (x_2, y_2). Can you see why?

Lets prove why this formula works.

Proof:
Consider a diagonal line AB. We notice that by drawing a horizontal line through A and a vertical line through B we get a right angled triangle. Therefore, we know by Pythagorean Theorem that $d^2 = a^2 + b^2$. Now we just need to figure out the lengths of a and b.
We can see that for the horizontal line a, the length is the total length from the y-axis to the intersection of a and b, minus the distance from the y-axis to point A. Algebraically, that is $x_2 - x_1$. Similarly, the length of the vertical line b is $y_2 - y_1$.



Substituting we get $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ as required.

Now consider a vertical and horizontal line, we have seen that they have lengths $x_2 - x_1$ and $y_2 - y_1$ respectively. We must show that this follows from our equation.
For a horizontal line, the y-values are always the same. Therefore $y_2 - y_1 = 0$, and in our equation we get $d = \sqrt{(x_2 - x_1)^2 + 0} = x_2 - x_1$ as required. Similarly, $d = \sqrt{0 + (y_2 - y_1)^2} = y_2 - y_1$ for a vertical line.
Thus for any line segment, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. $\square$

Slope
Slope is used to describe the steepness or gradient of a straight line. We represent slope by the letter m.
$\mathbf{m} = \frac{\mathbf{\Delta y}}{\mathbf{\Delta x}}$
Where Δ (delta) means the “change in”. So this reads as “the change in y over the change in x”.

Example: The line that passes through the points $(-3, 5)$ and $(3, 1)$ intersects the y-axis at the point $(0, k)$. Determine the value of k.

Solution:
First we will write out the general formula for the slope of a line.
$m = \frac{\Delta y}{\Delta x}$
We know that the line passes through the two points $(-3, 5)$ and $(3, 1)$. Thus, we can use the formula to find the slope of our line.
$m = \frac{1-5}{3--3}$
$m = \frac{-4}{6} = -\frac{2}{3}$
Now we know the slope of the line and can use our formula to work backwards to figure out the missing coordinate.
$\frac{k-1}{0-3} = -\frac{2}{3}$
Solving we get that $k = 3$.

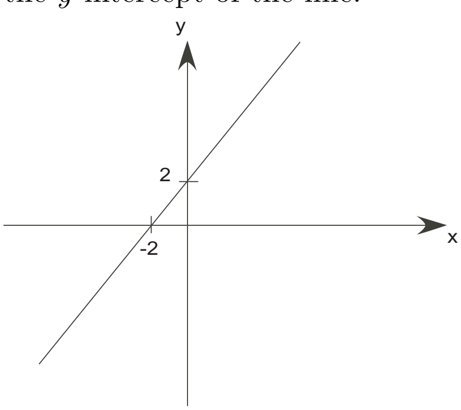
Parallel lines: If two lines are parallel they have the same slope.

Perpendicular line: If two lines are perpendicular (meet at an angle of 90°) then their slopes are the negative reciprocal's of one another.
ie) If l_1 and l_2 are two perpendicular lines and the slope of l_1 is $\frac{2}{3}$, then the slope of l_2 is $-\frac{3}{2}$.

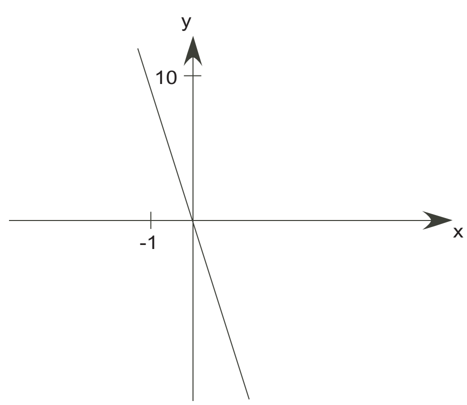
Extension:
We can only take the slope of a straight line. However, in Calculus we will use the slope to find the rate of change at certain points on a curve. The derivative, as you will learn, is closely related to the slope of the tangent line to a point on the curve.

Slope/Intercept Equation of a Line
Slope is a very important concept in mathematics. In fact, it is so important that we use it to help us define lines. However, slope alone is not enough to help us define a line. Many lines have the same slope. So to help us make a line unique we use the slope along with another point on the line. For simplicity, we use the y-intercept.

$\mathbf{y = mx + b}$ is the slope/intercept equation for a line. Where m represents the slope and b represents the y-intercept of the line.



y=x+2. Slope 1 and y-intercept 2



y=-10x. Slope -10 and y-intercept 0

Example: The line $y = ax + c$ is parallel to the line $y = 2x$ and passes through the point $(1, 5)$. What is the value of c?

Solution:
Since the line and $y = 2x$ are parallel we know that that they must have the same slope. Therefore, $a = 2$ and we now have $y = 2x + c$. But we also know that the point $(1, 5)$ lies on the line, and thus must satisfy the equation. Substituting into our equations we get $5 = 2(1) + c$. Solving we get $c = 3$.

Horizontal Lines: A line that is parallel to the x-axis, so the y value never changes. Horizontal lines have slope/intercept equation of the form $y = a$, where a is the y-intercept.
What is the slope of a Horizontal line? $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ but since the y value never changes, $y_2 = y_1$ and we get $\frac{0}{x_2 - x_1} = 0$.

Vertical Lines: A line that is parallel to the y-axis, so the x value never changes. Vertical lines are defined by the equation $x = a$, where a is the x -intercept.
What is the slope of a Vertical line? $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ but since the x value never changes, $x_2 = x_1$ and we get $\frac{y_2 - y_1}{0}$. Thus the slope of a vertical line is undefined.

Problems

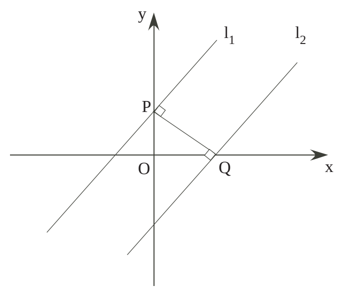
1. Show that $A(-4,-6)B(1,0)$ and $C(11,12)$ are collinear (all lie on a straight line) because $AB + BC = AC$

2. Determine how we can tell if a triangle given by three points is right, scalene, isosceles or equilateral.

3. The triangles given by the three points $A(6,4), B(4,-3)$ and $C(-2,y)$ is right at $\angle CBA$. Find y.

4. The line $y = \frac{-3}{4}x + 9$ crosses the x-axis at P, and the y-axis at Q. Point $T(r,s)$ is on the line segment PQ. If the area of $\triangle POQ$ is three times the area of $\triangle TOP$, then find the value of $r+s$.

5. The line l_1 has equation $y = mx+k$ and crosses the y-axis at P. Line l_2 crosses the x-axis at Q. If PQ is perpendicular to both lines, determine the y-intercept of line l_2 in terms of k and m,



6. Two perpendicular lines with x-intercepts -2 and 8 intersect at $(0,b)$. Determine all values of b.

7. Triangle ABC has vertices $A(-3,-1), B(1,1)$ and $C(5,-5)$. The median is drawn from A to side BC. Determine the equation of the line containing this median.

8. A triangle has vertices $A(-2,-11), B(10,5)$ and $C(12,3)$. Show that the circle with AB as a diameter passes through point C.

9. Determine the vertices of all squares that have one vertex at $P(25,0)$ and one side along the line $3x - 4y = 0$.

10. A point P is chosen on the line $y = 2x+3$ and a point Q is chosen on $y = -x+2$. If the midpoint M of the line segment PQ is $(2,5)$ calculate the coordinates of P and Q.