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Intermediate Math Circles
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Series

You are negotiating your next month's allowance with your parents. Your parents have given you two offers:

Offer 1: Your parents will give you \$10 the first day of the month, and on each subsequent day they will give you the amount they gave you the day before, plus 1 more dollar.

Offer 2: Your parents will give you \$0.01 the first day of the month, and on each subsequent day they will give you 2 times the amount they gave you the day before.

Questions:

1. For each offer, how much will you be given on day 1? day 2? day 3?

2. For each offer, what would be the total allowance amount in the first week?

3. Assuming there are 30 days in the month, which offer should you take?

What is a series?

A series is the sum of the terms in a sequence.

For the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, the corresponding series is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{31}{32}$

For the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ..., \frac{1}{2^{10}}$, the corresponding series is

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^{10}}=\frac{1023}{1024}$

Arithmetic Series:

An arithmetic series is the sum of the terms in an arithmetic sequence.

(Recall: An arithmetic sequence is a sequence where the difference between any consecutive pairs of numbers in the sequence is a constant. The general term of an arithmetic sequence is $a_n = a+(n-1)d$, where a is the first term in the sequence and d is the common difference.)

We are often interested in the **partial sum** of a series. The partial sum, S_n, is the sum of the first n terms.

For example, for the sequence: 1, 3, 5, 7, 9, ...

If $n = 7$, $S_7 = 1+3+5+7+9+11+13 = 49$

What is S_{151}?

Question:
In general, for any arithmetic series with first term a and common difference d, can we find a formula for S_n, the sum of the first n terms?

First, consider the series $1 + 2 + 3 + 4 + ... + 100$

(This is the arithmetic series with $a = 1$ and $d = 1$)

Can you think of a quick way to determine what this sum is?

A series that comes up frequently is $1 + 2 + 3 + 4 + ... + n$

Show that this series is equal to $\dfrac{n(n+1)}{2}$

Question:
In general, for any arithmetic series with first term a and common difference d, can we find a formula for S_n, the sum of the first n terms?

Let's use this same idea to come up with a formula for S_n, the sum of the first n terms of an arithmetic sequence with first term a and common difference d.

$S_n \;=\; a + (a+d) + (a+2d) + (a+3d) + ... + (a+(n-1)d)$

Write S_n twice, once forward and once backwards, and add:

$$\begin{array}{rlllll} S_n = & a & + (a+d) & + ... + (a+(n-2)d) & + (a+(n-1)d) \\ + \; S_n = & (a+(n-1)d) & + (a+(n-2)d) & + ... + (a+d) & + a \\ \hline 2S_n = & (2a+(n-1)d) & + (2a+(n-1)d) & + ... + (2a+(n-1)d) & + (2a+(n-1)d) \end{array}$$

$2S_n = n \times (2a + (n-1)d)$

$$S_n = \frac{n(2a+(n-1)d)}{2}$$

We can also re-arrange this to:

$$S_n = \frac{n(2a+(n-1)d)}{2} = \frac{n(a+a+(n-1)d)}{2} = \frac{n(a_1+a_n)}{2}$$

So we have two different forms for S_n:

$$S_n = \frac{n(2a+(n-1)d)}{2} \qquad \text{and} \qquad S_n = \frac{n(a_1+a_n)}{2}$$

Determine the sum of the first 20 terms of an arithmetic series in which

a) the first term is $a = 13$ and the common difference is $d = 7$

b) $a_1 = 31$ and $a_{20} = 175$

c) $a_5 = 142$ and $a_{15} = 12$

Geometric Series:

An geometric series is the sum of the terms in an geometric sequence.

(Recall: A geometric sequence is a sequence where there is the same ratio between any consecutive terms. The general term of a geometric sequence is $a_n = ar^{n-1}$, where a is the first term in the sequence and d is the common difference.)

Again, we are often interested in the **partial sum** of the series. The partial sum, S_n is the sum of the first n terms of the sequence.

For example, for the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$

If $n = 5$, $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{32} = \frac{31}{32}$

What is S_{151}?

Question:
In general, for any geometric series with first term a and common ratio r, is there a formula for S_n, the sum of the first n terms?

Let's use the same idea we used to to come up with a formula for the sum of the first n terms of any arithmetic sequence.

Calculate S_n for any geometric sequence with first term a and common ratio r:

$$S_n \;=\; a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Use a slight modification of Gauss' method:

$$\begin{array}{rl} S_n = & a + ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} \\ -rS_n = & -(\quad ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} + ar^n) \\ \hline (1-r)S_n = & a - ar^n = a(1 - r^n) \end{array}$$

So, $S_n = \dfrac{a(1-r^n)}{(1-r)}$, if $r \neq 1$ (What is S_n if r = 1?)

We can also re-arrange this to:

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a - ar^n}{(1-r)} = \frac{(a_1 + a_{n+1})}{1-r}$$

So we have two different forms for S_n:

$$S_n = \frac{a(1-r^n)}{(1-r)} \qquad \text{and} \qquad S_n = \frac{(a_1 - a_{n+1})}{1-r}$$

Determine the sum of the first seven terms of the geometric series in which

a) $a_1 = 3$ and $r = 5$

b) $a_1 = 12$ and $a_2 = 6$

c) $a_5 = 5$ and $a_8 = -30$

What is the value of $-3 + 12 - 48 + ... + 786432$?

More Problems:

1. Determine the sum of the first 10 terms of the arithmetic series with $a_1 = 114$ and $a_2 = 99$
2. The 10th term of an arithmetic series is 14 and the sum of the first 20 terms is 210. Determine the 25th term.
3. Bricks are stacked in 16 rows such that each row has a fixed number of bricks less than the row below it. The top row has 7 bricks and the bottom row has 52 bricks. How many bricks are in the stack?
4. Natalie is training to run a marathon. The first week she runs 5km each day. The next week she runs 7km each day. During each successive week, each day she runs 2km farther than she ran the days of the previous week. If she runs for five days each week, what total distance will Natalie run in a 10 week training session?
5. The first triangular number is 1, the second is 3, the third is 6, the fourth is 10, and the nth triangular number equals $1 + 2 + 3 + ... + (n - 1) + n$.
   a) Calculate the 10th and 24th triangular numbers.
   b) Prove that the sum of any three consecutive triangular numbers is always 1 more than three times the middle of these three triangular numbers.
   c) Why do you think these numbers are called "triangular"?
6. The sum of fifty consecutive *even* integers is 3250. What is the largest of these integers?
7. Back to the allowance problem. What is the total amount that you will receive in 30 days from Offer 1? What about Offer 2? Which offer should you take?
8. Determine the sum of the first 7 terms of a geometric series with $a_1 = 11$ and $a_7 = 704$.
9. A ball is dropped from a height of 4 m and bounces on the ground. A the top of each bounce, the ball reaches 75% of its previous height. Calculate the total distance travelled by the ball when it hits the ground for the 5th time.
10. At exactly one o'clock two bacteria are placed in a petri dish. One minute later there are four bacteria. In another minute there are eight bacteria, etc. At exactly two o'clock there are 400 mL of bacteria. At what time was there 100 mL of bacteria?