University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

# Intermediate Math Circles
# November 12, 2008
# Geometry III

## Geometry

Polygons

Definitions:
A *polygon* is the 2-D figure created by having a sequence of points, and connecting each point to the next to form a cycle.
A polygon is *equiangular* if all of its angles are equal.
A polygon is *equilateral* if all of its sides are equal.
A polygon is *regular* if it is equiangular and equilateral. A regular polygon with $n$ sides is sometimes denoted by $\{n\}$.

Polyhedra

Definitions:
*Polyhedra* are 3-D figures with surfaces composed of polygons.
Polyhedra have polygonal *faces*, straight *edges* where the sides of two faces meet, and *vertices* where the corners of polygons meet.
A polyhedron is called *convex*, if for every two points chosen on the surface of the polyhedron, the line segment connecting them is completely contained within the polyhedron.
A polyhedron is called *regular* if all of its faces are identical regular polygons, and the same number of faces meet at each vertex. We sometimes denote these polyhedra by $\{p, q\}$ where the faces are regular $p$-gon's and $q$ faces meet at each vertex.

Theorem(Euler's Formula):
In a convex polyhedron, with $F$ faces, $E$ edges, and $V$ vertices,
$F - E + V = 2$.

Definition:
The *platonic solids* are the polyhedra which are both convex and regular.

Theorem:
There are only 5 platonic solids.

Proof:
Since each face has $p$ edges, and each edge is shared by two faces, we have $pF = 2E$ or $F = \frac{2E}{p}$. Since there are $q$ edges meeting at each vertex, and each edge joins two vertices, we have $qV = 2E$ or $V = \frac{2E}{q}$.
Plugging these values into Euler's Formula we get $\frac{2E}{p} - E + \frac{2E}{q} = 2$ or $\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}$.
Now, since polygons must have at least 3 sides, we have $p \geq 3$ and $q \geq 3$. However if $p > 3$ and $q > 3$, $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < \frac{1}{2} + \frac{1}{E}$. Therefore $p = 3$ or $q = 3$.
If $p = 3$, then $q$ may equal 3, 4, or 5. If $p = 3, q = 3$ then $E = 6$, if $p = 3, q = 4$ then $E = 12$ and if $p = 3, q = 5$ then $E = 30$.
Now, if $q = 3$, then $p$ may equal 3, 4, or 5. If $p = 3, q = 3$ we have $E = 6$ as above. If $p = 4, q = 3$ then $E = 12$ and if $p = 5, q = 3$ then $E = 30$.

These are the only possible combinations, so the 5 platonic solids are $\{3, 3\}$ which is a tetrahedron, $\{3, 4\}$ which is an octahedron, $\{3, 5\}$ which is an icosahedron, $\{4, 3\}$ which is a cube, and $\{5, 3\}$ which is a dodecahedron.

## Weaving Platonic Solids

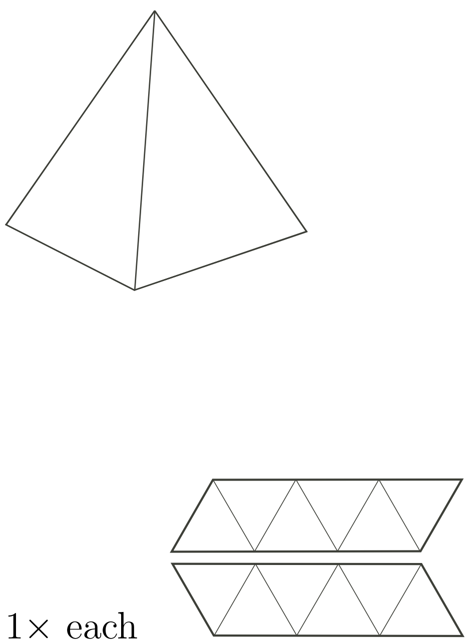
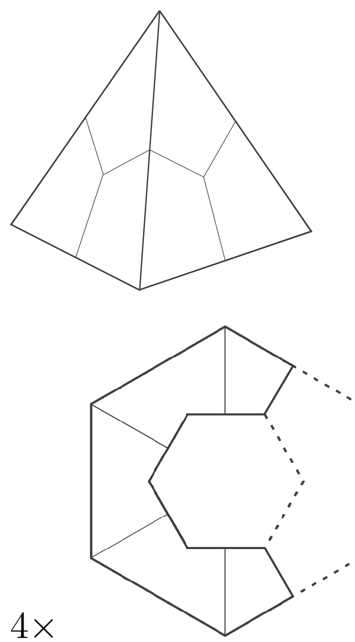
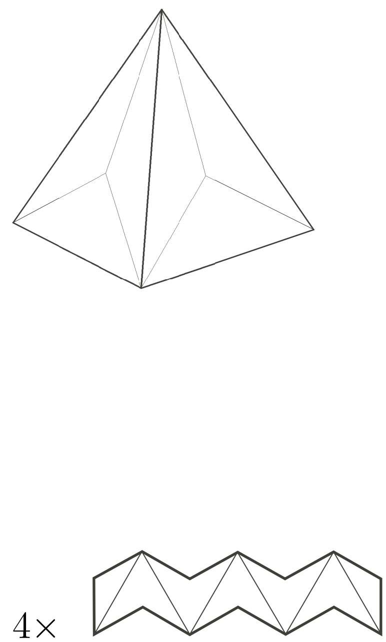
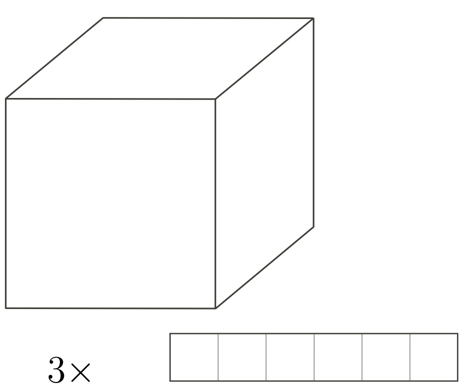
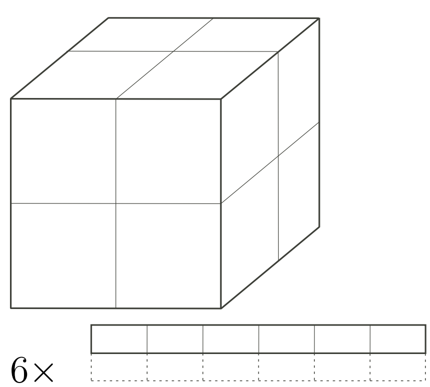
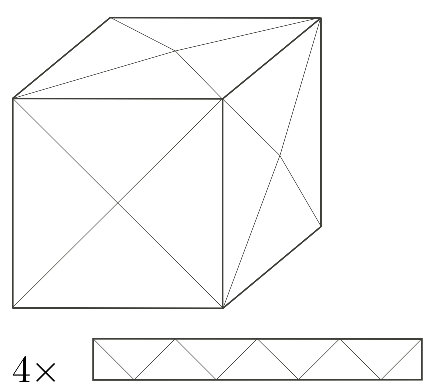
If you want to create your own models of the platonic solids, you can do it by weaving paper.

The tables below show three ways each to weave the five platonic solids. The upper picture shows what the completed solid will look like. The picture below indicates the number and shape of paper strips you will need to weave it.

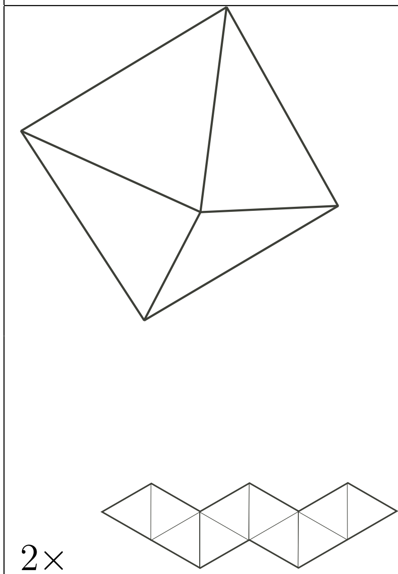
Draw these strips on sheets of coloured paper. Using many different colours results in a more interesting finished solid, as well as a less confusing weaving process. The heavy black outlines are the lines you will cut along, and the inner black lines are fold lines. The dotted lines are only included to make the shape of the strip you need to draw clearer.

To begin weaving, take the number of strips you need to complete one face (for example, 3 strips for the octahedron). Weave these strips together, each of them overlapping one other strip, so that you have one face formed, and the ends of the strips are curling towards you. Next, add one strip at a time, weaving it through the existing strips to form the adjacent faces.

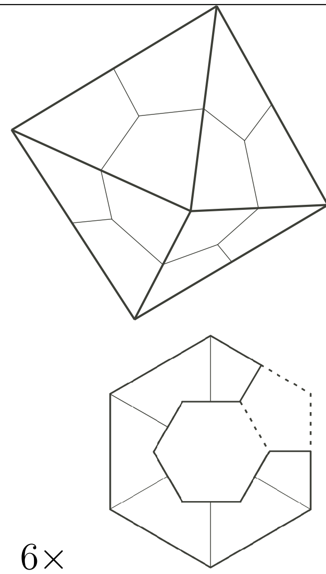
When a strip is completely woven into the solid, you should be able to tuck its ends under other strips. There should be no need for glue, tape, or staples to hold the solid together.

| Tetrahedron | | |
|---|---|---|
|  1× each |  4× |  4× |
| Cube | | |
|  3× |  6× |  4× |

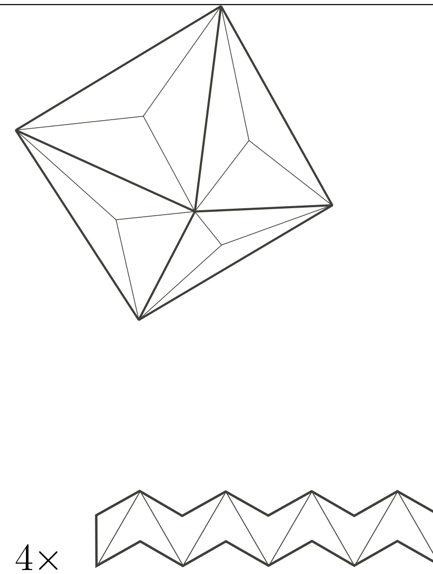
## Octahedron



2×

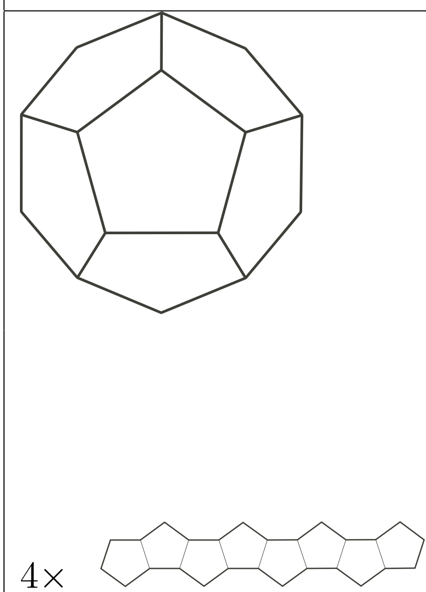


6×

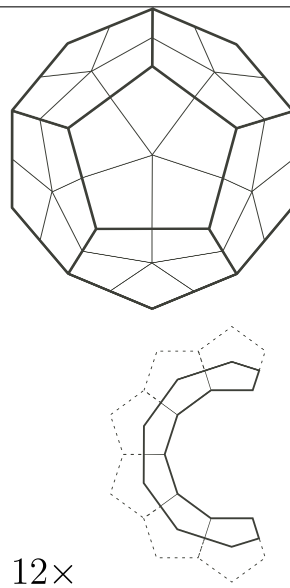


4×

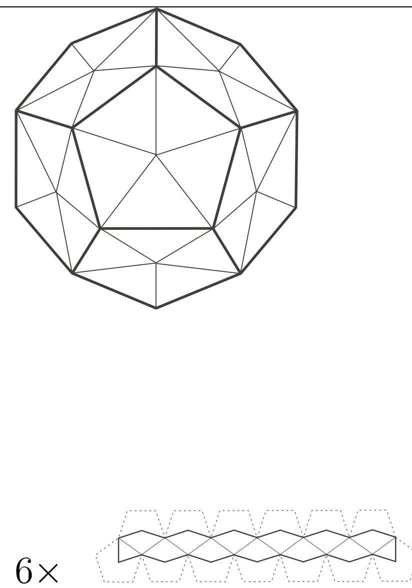
## Dodecahedron



4×

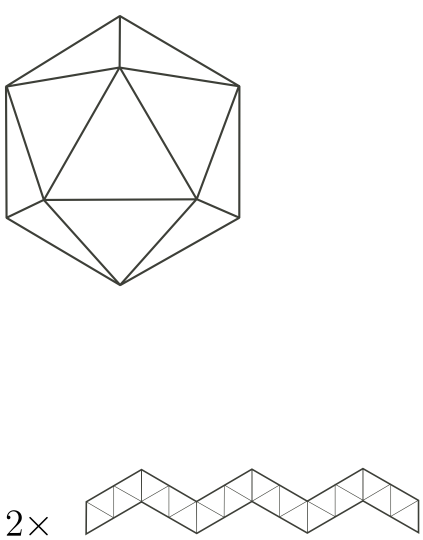


12×

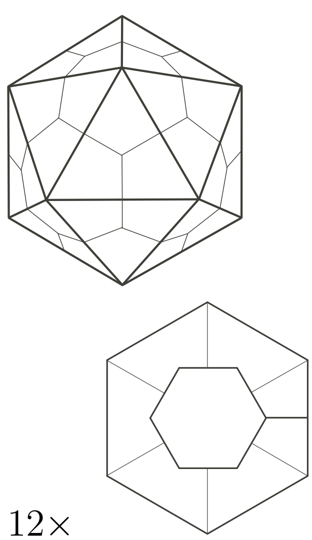


6×

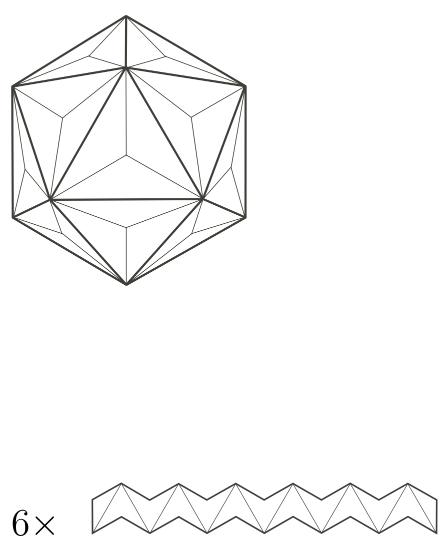
## Icosahedron



2×



12×



6×

## Problem Set

1. Is it possible to have a polygon that is equilateral, but not equiangular? Is it possible to have a polygon that is equiangular, but not equilateral? Find two quadrilaterals as examples.

2. How many faces, edges, and vertices do each of the platonic solids have?

3. The cuboctahedron has 8 faces that are equilateral triangles, and 6 faces that are squares. How many vertices does this shape have? Why isn't it considered regular?

4. An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

5. A truncated icosahedron has 60 vertices, with 3 edges meeting at each vertice. How many edges and faces does this solid have?

6. Of the truncated icosahedron's faces, 12 are regular $p$-gons and the remainder are regular $q$-gons. What are the two possible ordered pairs $(p, q)$? Only one of these two ordered pairs represents the truncated icosahedron. What polyhedron does the other ordered pair represent?