University of Waterloo
Faculty of Mathematics

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# Intermediate Math Circles
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# Probability I

## Probability

Sets

Definition:
A *set* is simply a collection of objects. The objects can be anything, letters, numbers, people...
Sets are often labeled using upper case letters.

Example:
$A = \{1, 2, 3\}$. $A$ is the set containing the positive integers 1, 2, and 3. The order of the *elements* or *members* of the set does not matter.

Definitions:
A *subset* of a set is another set which contains all or some of the elements of a set, or nothing. The set with nothing in it is called the *empty set* or the *null set*, and is considered to be a subset of every set.

Example:
The subsets of the set $A$ given above are:

$$\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{\ \}$$

There is 1 subset of $A$ with no elements, 3 with one element, 3 with two elements, and 1 with three elements, a total of 8 subsets.

In general, if a set has $k$ elements in it, there are $2^k$ subsets as there are two choices for each element, put it in the subset or leave it out.

Definition:
The *cardinality* of a set $A$, denoted $n(A)$, is simply the number of elements in the set.

Example:
For the set $A$ given above, $n(A) = 3$.

Probability

Definitions:
The set of all possible outcomes for an activity, called an *experiment*, is called the *sample space* of the experiment.

Examples:
For tossing a coin, the sample space is $S = \{H, T\}$, denoting Heads and Tails.
For rolling a standard die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
For rolling a pair of standard dice, the sample space consists of 36 ordered pairs of values, each of which come from the set $\{1, 2, 3, 4, 5, 6\}$. The sample space for rolling two dice is:

$$\begin{array}{llllll} \{(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2,) & (6,3), & (6,4), & (6,5), & (6,6)\} \end{array}$$

Definition:
An *event* is a subset of the sample space and the probability of any event, $E$, is defined to be $p(E) = \dfrac{n(E)}{n(S)}$.

Example:
Consider the experiment of rolling a die. Define event $E$ to be rolling a 1, 3, or 5, ie $E = \{1, 3, 5\}$, and define event $F$ to be rolling a 1 or a 2, ie $F = \{1, 2\}$.

$$p(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2} \text{ and } p(F) = \frac{n(F)}{n(S)} = \frac{2}{6} = \frac{1}{3}.$$

Definitions:
The event ($E$ or $F$) is defined as either an outcome for $E$ occurs or an outcome for $F$ occurs. The subset which represents ($E$ or $F$) is the *union* of the sets representing $E$ and $F$.

The event ($E$ and $F$) is defined an outcome for $E$ occurs and an outcome for $F$ occurs. The subset which represents ($E$ and $F$) is the *intersection* of the sets representing $E$ and $F$.

Example:
In the situation above where you are rolling a die, $E \cup F = \{1, 2, 3, 5\}$ and $E \cap F = \{1\}$.
Therefore $p(E \cap F) = \frac{1}{6}$ and $p(E \cup F) = \frac{4}{6} = \frac{2}{3}$

Note that $p(E \cup F) = p(E) + p(F) - p(E \cap F)$.

## Problem Set

1. Let $A$ be the set of all prime numbers between 6 and 22? What are all subsets of $A$ with a cardinality of 2?

2. The corner store in your neighbourhood sells 5 different ice cream flavours. If you can get a banana split with any combination of any number of ice cream flavours, from no ice cream to all 5, how many different combinations could you make?

3. Let the set $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the set $D = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$. What is $C \cup D$? What is $C \cap D$?

4. You roll two standard dice and take the sum of the top numbers. What is the probability that this sum will be less than 7?

5. You are given a well-shuffled deck of 52 cards and asked to choose one card at random.

   (a) What is the probability that it will be a diamond?

   (b) What is the probability that it will be a face card (face cards are Jacks, Queens, Kings, and Aces)?

   (c) What is the probability that it will be both a diamond *and* a face card? What is the probability that it will be a diamond *or* a face card.

6. There are 40 balls in a hat, each of which are either red, blue, or green, and have a number from 1 to 5 written on them. If the probability of drawing a red ball with a "2" on it is $\frac{1}{10}$, the probability of drawing a ball that is either red or has a "2" on it or both is $\frac{1}{4}$, and there are 36 red balls in the hat, what is the probability of drawing a ball with a "2" on it?