University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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Probability II

Probability

Counting

Definition:
We define a *factorial* to be $n! = n(n-1)(n-2)(n-3)\ldots(2)(1)$ for n a positive integer. By definition, $0! = 1$.

If we have a group of 32 people from which we are going to select a President, a Vice-President, and a Secretary-of-State, we can do this in $32 \times 31 \times 30 = 29760$ ways. That is, we have 32 choices for President, for each of those we have 31 choices for Vice-President, and for each of these pairs there are 30 choices for the Secretary-of-State.

We can develop a formula for this as follows:

$$32(31)(30) = 32(31)(30) \times \frac{29!}{29!} = \frac{32(31)(30) \times 29!}{29!} = \frac{32!}{29!}$$

Here we are looking at ordered triples (sets of three) taken from the set of 32 people. We can think of the above fraction as: $\dfrac{32!}{(32-3)!}$.

This leads us to the generalization, if an ordered subset of size k is to be taken from a set of size n, there are $P(n,k) = \dfrac{n!}{(n-k)!}$ ways to select the ordered set. These ordered set are called *permutations of n things taken k at a time.*

We later decide to select a committee of three people rather than a President, a Vice-President, and a Secretary-of-State, from the 32 people. Since the order of selection no longer matters, we have fewer committees than the ordered triples we had before.

For example; if Allan, Betty, and Clara were three of the members of the class, the number 29760 computed before would include each of the following arrangements:

President	Vice-President	Secretary-of-State
Allan	Betty	Clara
Allan	Clara	Betty
Clara	Allan	Betty
Clara	Betty	Allan
Betty	Clara	Allan
Betty	Allan	Clara

Since the order no longer matters, these six triples only count as one. Thus we can divide the number we had before by 6 (notice that this is the number of arrangements of the three people, 3!). Thus there are $\frac{29760}{6} = 4960$ selections of 3 people from the 32.

Generalizing the number of ways to select k items from a set of n items is

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

These unordered selections are called *combinations of n things taken k at a time*, or more simply the number $C(n,k)$ is the number of ways to choose k things from a set of n things.

Problem Set

1. In the game *Risk*, dice are rolled to simulate battles. Ties go to the defender. Determine the probability that the defender wins on a roll when he rolls a single die against a roll of three dice by the attacker.

2. The numbers 1, 2, 3, 4, and 5 are written on slips of paper, and two slips are drawn at random without replacement. determine the probability of each of the following.

   (a) The sum of the numbers is 9.

   (b) The sum of the numbers is 5 or less.

   (c) The first number is 2 or the sum is 6.

   (d) Both numbers are even.

   (e) One of the numbers is even or greater than 3.

   (f) The sum is 5 or the second number is 2.

3. Given the equally-likely sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
   and the events $E = \{4, 8, 12\}$
   $F = \{2, 4, 6, 8, 10, 12\}$
   $G = \{3, 6, 9, 12\}$
   $H = \{2, 3, 5, 6, 8, 9, 11, 12\}$
   find each of the following:

   (a) Each of $p(E), p(F), p(G)$, and $p(H)$

   (b) Each of $p(E \cap F)$ and $p(E \cup F)$

   (c) Each of $p(E \cap G)$, $p(G \cap H)$, and $p(H \cap G)$

   (d) Each of $p(E \cap F \cap G)$ and $p(E \cup F \cup G)$

4. A marble is drawn from a box containing 3 yellow, 4 white, and 7 blue marbles.

   (a) Find the probability of selecting a yellow marble.

   (b) Find the probability of selecting a white marble.

   (c) Find the probability of selecting a blue marble.

   (d) Find the probability of **not** selecting a yellow marble.

   (e) Find the probability of selecting a yellow or a blue marble.

   (f) If you select 2 marbles at random, what is the probability they are the same colour?

   (g) If you select 2 marbles at random, what is the probability they are different colours?

5. There are three dice on a table. Die A has the numbers 1, 1, 5, 5, 5, 5 on its faces. Die B has the number 3 on every face, and Die C has the numbers 2, 2, 2, 2, 6, 6 on its faces. Each of two players choose a die. They each roll their die and the higher number wins.

   (a) If dice A and B are chosen, what is the probability that the player with Die A wins?

   (b) If dice B and C are chosen, what is the probability that the player with Die B wins?

   (c) If dice C and A are chosen, what is the probability that the player with Die C wins?

   Summarize your results.

6. The integers 1, 2, 3, 4, 5, 6, and 7 are arranged at random to form a seven-digit number.

   (a) Determine the probability that the number formed is odd.

   (b) Determine the probability that the number formed is less than 3 200 000.

   (c) Determine the probability that the number formed is divisible by 4.

7. Lotto 649 requires a player to pick 6 numbers(without repetition) from the integers 1 to 49. In how many ways can this be done?

8. Calculate the number of bridge hands which contain the following:

   (a) 5 clubs and 5 hearts

   (b) 4 clubs and 3 cards in each of the other suits

   (c) seven cards of one suit.