University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

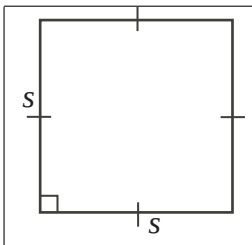
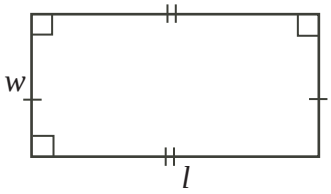
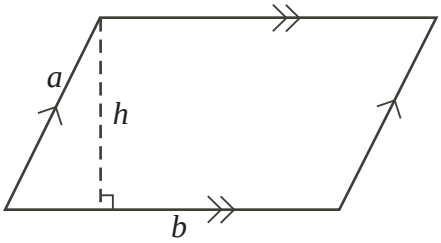
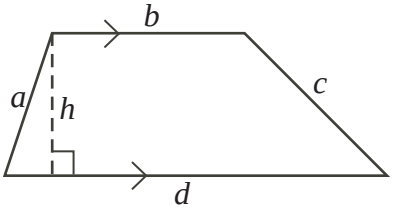
Intermediate Math Circles
November 5, 2008
Geometry II

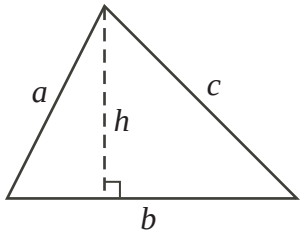
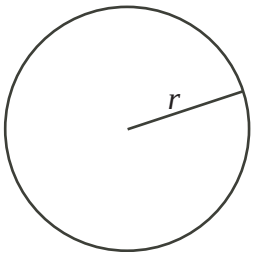
Geometry

2-D Figures

Two-dimensional shapes have a perimeter and an area.
Perimeter is the length of the outline of a shape.
Area is the surface the shape covers.

The table below gives formulas for the perimeter P and area A of some common 2-D figures.

 $P = 4s$ $A = s^2$	 $P = 2(l + w)$ $A = lw$
 $P = 2(a + b)$ $A = bh$	 $P = a + b + c + d$ $A = h\frac{b+d}{2}$

	
$P = a + b + c$ $A = \frac{bh}{2}$	$P = 2\pi r$ $A = \pi r^2$

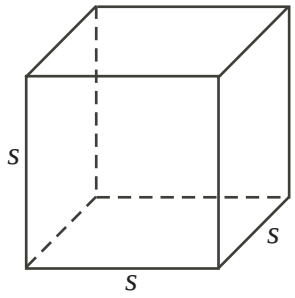
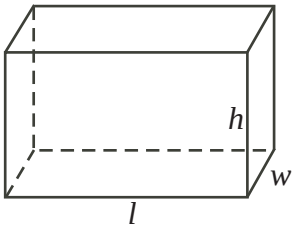
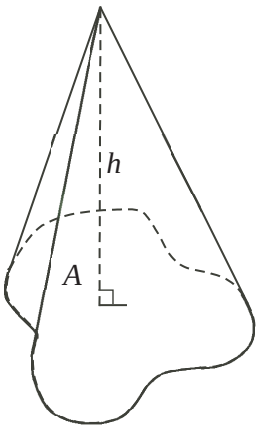
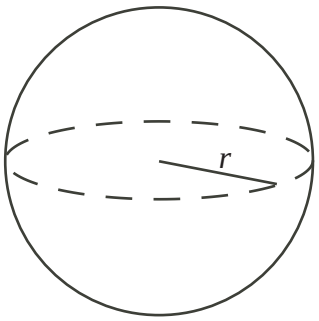
Example:
A rectangle has dimensions 24cm $\times$ 15cm. If the longer side is decreased by 6cm, by how many centimetres must the other side be increased for the new rectangle to have the same area as the original?

Solution:
The original rectangle has area $A = 24 \times 15 = 360\text{cm}^2$.
The new rectangle has length $24 - 6 = 18$cm, and width $(15 + x)$cm.
Therefore, its area is $A = 18(15 + x) = 270 + 18x$. However, this area is the same as the original so $360 = 270 + 18x$ or $90 = 18x$ and so $x = 5$cm.
Therefore, the shorter side must be increased by 5cm to keep the area the same.

3-D Figures

Three-dimensional figures have a surface area and a volume.
Surface area is the two-dimensional area the surface of a 3-D figure takes up.
Volume is the amount of space a figure takes up.

The table below gives formulas for the surface area SA and volume V of some 3-D figures.

 $SA = 6s^2$ $V = s^3$	 $SA = 2(lw + lh + wh)$ $V = lwh$
 Let A be the area of the base. $V = \frac{1}{3}Ah$	 $SA = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$

Note that the triangle-based pyramid, square-based pyramid, and cone are all special cases of the formula in the bottom left of the table. The bases of these solids would be a triangle, a square, and a circle, respectively.

Example:
A cube with side length 10 rests inside a sphere, so that each of the 8 vertices of the cube touch the surface of the sphere. What is the volume of this sphere?

Solution:
Since all vertices of the cube are touching the sphere, the diagonal of the cube will be equal to the diameter of the sphere. The diameter of the sphere is therefore
$d = \sqrt{s^2 + s^2 + s^2}$
$d = \sqrt{3s^2}$
$d = s\sqrt{3}$
$d = 10\sqrt{3}$

Therefore, since $d = 2r$, $r = 5\sqrt{3}$. So, the volume of the sphere is
$V = \frac{4}{3}\pi r^3$
$V = \frac{4}{3}\pi(5\sqrt{3})^3$
$V = \frac{4}{3}\pi(375\sqrt{3})$
$V = 500\sqrt{3}\pi$

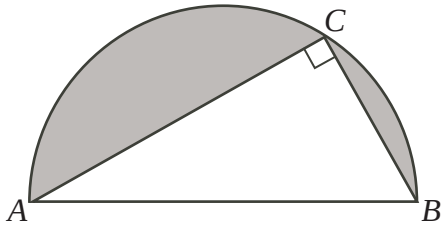
Problem Set

1. A piece of string, 40cm long, is formed into a circle with the ends of the string touching each other. What is the radius of the circle?

2. A square of side length $2n + 1$ has another square of side length $2n$ inside it. What is the area between the two squares in terms of n?

3. The lengths, in inches, of the three edges meeting at each corner of a rectangular prism are 1, 2 and 3. What is the length of the diagonal of this prism?

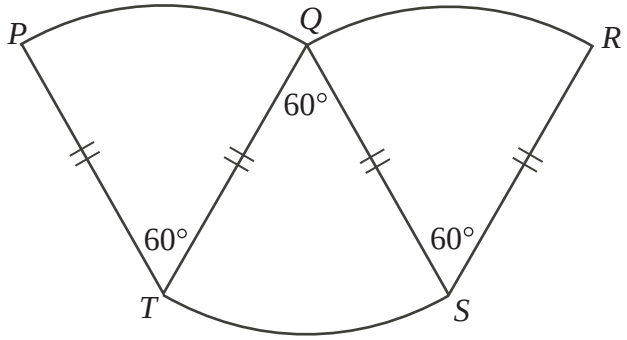
4. In the diagram, AB is the diameter of the semicircle. If $AC = 8$, $CB = 6$, and $\angle ACB = 90°$, what is the area of the shaded portion?



5. A $3 \times 3 \times 3$ cube is painted red and is then cut into 27 unit cubes. How many of these $1 \times 1 \times 1$ cubes have paint on exactly two of their faces?

6. A wire 60cm in length is cut into two parts in the ratio 2 : 1. Each part is bent to form a square. What is the total area of the two squares?

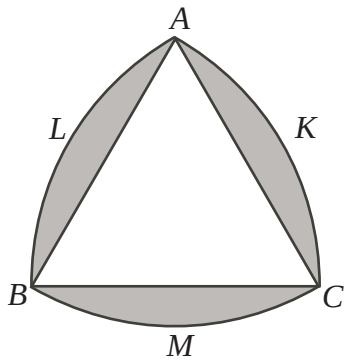
7. In the diagram, circular arcs PQ, QR and ST have centres T, S and Q respectively. If $PT = QT = QS = SR = 1$, then what is the perimeter of figure $PQRST$?



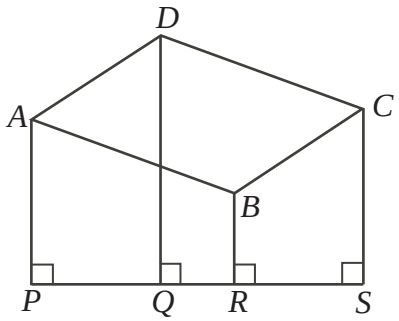
8. An open box is made from a square piece of tin by removing a 6 inch square from each corner and folding the sides upwards. If the volume of the box is 864 cubic inches, find the area of the original square.

9. The tip of a straight reed growing in the centre of a pond 8 feet in diameter reaches one foot above the water. When the reed is pulled over to the edge, with its bottom fixed, the tip just touches the edge of the pond. What is the depth of the pond in feet?

10. In the diagram, ABC is an equilateral triangle having side length 2, and ALB, BMC and AKC are arcs of circles having centres C, A and B, respectively. Find the total area of the shaded regions in the diagram.



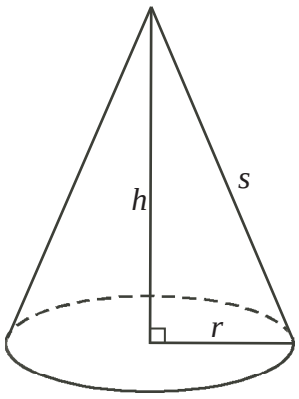
11. $ABCD$ is a parallelogram in which $AP = 12$, $DQ = 16$, $CS = 10$, $PQ = 5$, and $QR = 2$. Find the area of figure $BRSC$.



12. A rectangular house which measures 20m $\times$10m has an outside electrical outlet at a corner of the house. An electric mower, connected by a cord to the outlet, can reach a maximum distance of 15m. What is the largest area of lawn which can be cut?

13. A circle of radius r is rolled around the outside of a rectangle of perimeter p, always maintaining contact with the rectangle. What is the distance travelled by the centre of the circle, when the circle has travelled once around the rectangle?

14. A paper cone, when cut along its slant height s, and flattened out, forms a semi-circle of radius 10cm. Find the height h of the cone.



15. A manufacturer sells clear plastic tape on a spool with radius 1cm. The tape is 0.02cm thick and 1.5cm wide. The combined radius of the spool and the tape is 3cm. What is the best approximation of the length of the tape on this spool, in metres, to one decimal place?