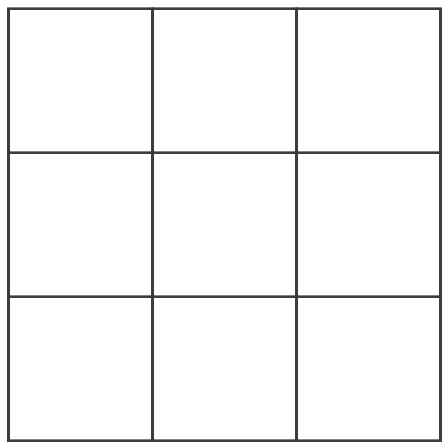
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
October 15, 2008
Number Theory II

Opening Problem

Take a 3×3 grid:



Determine all ways that you can place X's in the squares so that

- no three X's are in a straight line, and
- any additional X that is added will create a straight line of three X's.

Alphonse and Beryl play a game by alternately putting a penny in an empty square of a 3 by 3 grid. The loser is the first to put a coin in that makes three-in-a-line. If Alphonse goes first, who has a winning strategy?

Number Theory

Why are prime numbers important?

They're important because every positive integer larger than 1 is the product of primes.

For example, $45 = 3 \times 3 \times 5$.

Should this be the same or different from $45 = 3 \times 5 \times 3$?

Let's say that these are the same, since they use the same primes and the order of multiplication is irrelevant.

In fact, every positive integer larger than 1 is the product of prime numbers in a unique way (aside from reordering the prime numbers).

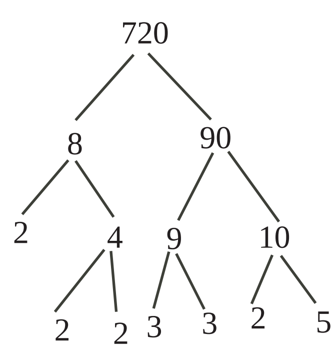
Prime numbers also have very significant "real life" applications.

Why is 1 not considered a prime number?

It is not prime because it would throw off the uniqueness of prime factorization.
If 1 was prime, then $15 = 1 \times 1 \times 1 \times 3 \times 5$ and $15 = 1 \times 1 \times 3 \times 5$ would be different prime factorizations, and so no prime factorizations would be unique at all!

How would we find the prime factorization of a number?

We could make a factor tree:



So $720 = 2^4 3^2 5^1$.

Problems:

1. Find the prime factorization of 10 080.
2. $20! = (20)(19)(18)(17)(16)\cdots(4)(3)(2)(1)$.
   Find the prime factorization of 20!.

It might be useful to find faster ways of finding factors other than by dividing. Let's talk about tests for divisibility:

- Divisibility by 2?
  Check if units digit is even.
- Divisibility by 3?
  Find the sum of the digits. Check if this sum is divisible by 3.

  Why is this better than just dividing by 3?
  It reduces the problem to a smaller problem.

  Why does this work?

  Let's suppose our number was $n = 23487$.
  Then

  $$\begin{aligned} n &= 2(10000) + 3(1000) + 4(100) + 8(10) + 7 \\ &= 2(9999) + 2 + 3(999) + 3 + 4(99) + 4 + 8(9) + 8 + 7 \\ &= [2(9999) + 3(999) + 4(99) + 8(9)] + [2 + 3 + 4 + 8 + 7] \end{aligned}$$

  The expression in the first set of brackets is a multiple of 3, so n is a multiple of 3 as long as the expression in the second brackets is.
- Divisibility by 4?
  Check if integer formed by last two digits is a multiple of 4.
- Divisibility by 5?
  Check if the units digit is 0 or 5.
- Divisibility by 6?
  Check if the number is divisible by 2 and by 3.
- Divisibility by 8?
  Check if integer formed by last three digits is a multiple of 8.
- Divisibility by 9?
  Find the sum of the digits. Check if this sum is divisible by 9.

- Divisibility by 10?
  Check if the units digit is 0.
- Divisibility by 11?
  Find the *alternating sum* of the digits. Check if this alternating sum is divisible by 11. (If $n = 15764$, the alternating sum is $1 - 5 + 7 - 6 + 4 = 1$.)
- Divisibility by 12?
  Check if the number is divisible by 4 and 3.

What about divisibility by 7?

You could divide the number by 7, but there is another way. Start with the integer, remove the units digit and subtract 2 times this units digit from the number that is left.
Repeat until you get a small number and check if this is divisible by 7.

Example:

$$\begin{aligned} 18767 &\rightarrow 1876 - 2(7) = 1862 \\ &\rightarrow 186 - 2(2) = 182 \\ &\rightarrow 18 - 2(2) = 14 \end{aligned}$$

which is divisible by 7. Therefore, 18767 is divisible by 7.

Problem:
Is 1672564 divisible by 7? Use this method!

What about divisibility by 13?

Use the method for 7, but add 4 times the units digit instead of subtracting 2 times the units digit.

Problem:
Is 161174 divisible by 13?

Problem Set

1. Every 14 days Bill mows the lawn, and every 8 days Alice weeds the garden. On May 1st Bill was mowing the lawn, and Alice was weeding the garden. What is the next day on which they will both be doing these chores?

2. Four teams play a game, and the product of their scores is 491400. If their scores are each between 20 and 30, what is the sum of their scores?

3. How many odd factors does 22176 have?

4. What is the smallest value of x such that $4725x$ is a perfect square?

5. If the six digit number 2a60b4 is divisible by 8 and 9, what are all possible ordered pairs of digits (a, b)?

6. What is the largest power of 18 that divides 100! $(100! = (100)(99)(98)(97)\cdots(3)(2)(1))$?

7. How many possible ordered pairs (a, b) are there such that the 4-digit positive integer abba, that is $1000a + 100b + 10b + a$, is divisible by 33?