University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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Number Theory III

Opening Problem

Carolyn and Paul are playing a game starting with a list of the integers 1 to n. The rules of the game are:

- Carolyn always has the first turn.
- Carolyn and Paul alternate turns.
- On each of her turns, Carolyn must remove one number from the list such that this number has at least one positive divisor other than itself remaining in the list.
- On each of his turns, Paul must remove from the list all of the positive divisors of the number that Carolyn has just removed.
- If Carolyn cannot remove any more numbers, then Paul removes the rest of the numbers.

For example, if $n = 6$, a possible sequence of moves is shown in this chart:

Player	Number(s) removed	Number(s) remaining	Notes
Carolyn	4	1, 2, 3, 5, 6	
Paul	1, 2	3, 5, 6	
Carolyn	6	3, 5	She could not remove 3 or 5
Paul	3	5	
Carolyn	None	5	Carolyn cannot remove any number
Paul	5	None	

In this example, the sum of the numbers removed by Carolyn is $4 + 6 = 10$ and the sum of the numbers removed by Paul is $1 + 2 + 3 + 5 = 11$.

(a) Find a partner. Play the game starting with $n = 12$ four times, with each of you going first twice. What was the largest score that the first player ever got?

(b) Suppose that $n = 6$ and Carolyn removes the integer 2 on her first turn. Determine the sum of the numbers that Carolyn removes and the sum of the numbers that Paul removes.

(c) Suppose that $n = 9$ and Carolyn removes the integer 6 on her first turn. What integer(s) can Carolyn remove on her second turn?

(d) How many prime numbers can Carolyn remove over the course of a game?

(e) If $n = 10$, determine Carolyn's maximum possible final sum. Prove that this sum is her maximum possible sum.

(f) If $n = 14$, prove that Carolyn cannot remove 7 numbers.

Number Theory

Here is a summary of tests for divisibility:

Divisibility by	Test
2	Last digit even
3	Sum of digits divisible by 3
4	# formed by last two digits divisible by 4
5	Last digit 0 or 5
6	Divisible by 2 and 3
7	"Subtract 2 times last digit from rest" algorithm
8	# formed by last three digits divisible by 8
9	Sum of digits divisible by 9
10	Last digit 0
11	Alternating sum digits divisible by 11
12	Divisible by 3 and 4
13	"Add 4 times last digit to rest" algorithm
14	Divisible by 2 and 7
15	Divisible by 3 and 5

Problems:

1. Which of the integers from 2 to 15 are factors of 182754184?
2. Which of the integers from 2 to 15 are factors of 234876153912120?

Why do we we care about being able to do this without a calculator?

Problem Set

1. A 4-digit number was written: 86□□ with the last two digits missing. If the complete number is exactly divisible by 3, by 4 and by 5, what are the missing digits?

2. If I add up all of the different prime factors of 1998, what answer do I get?

3. What is the sum of all of the prime numbers less than 25?

4. The 8-digit number 1234□678 is a multiple of 11. What digit is represented by □?

5. All except four of the nine numbers from 11 to 19 can be put in a single sequence "16, 18, 15, 12, 14" where each successive pair (such as 12 and 14, or 18 and 15) has highest common factor greater than 1. If you make the longest possible sequence like this using as many as possible of the nine numbers from 111 to 119, how many numbers will be left out?

6. The two-digit by two-digit multiplication below has lots of gaps, but most them can be filled in by logic. Do this!

$$\begin{array}{rrrr}
 & & 4 & _ \\
 & \times & _ & _ \\
\hline
 & _ & 8 & _ \\
 & 8 & _ & 0 \\
\hline
_ & _ & 4 & _
\end{array}$$

7. Mr. Brown teaches in a high school. The product of the ages of his students is 15 231 236 267 520. What are the ages of his students?

8. A nine-digit number uses each of the non-zero digits 1 to 9 exactly once. The number itself has to be divisible by 9. Also, the 8-digit number obtained by removing the last digit is evenly divisible by 8, the 7-digit number obtained by removing the last two digits is evenly divisible by 7, and so on. What is the number?

9. If the number $A364057B$ is divisible by 99, find the digits A and B.

10. Given any 16 composite integers less than 2500, at least two will have a prime factor in common. Explain why this is true.

11. Show that from any five integers, not necessarily distinct, one can always choose three of these integers whose sum is divisible by 3.

12. You have three piles of stones, containing 5, 49 and 51 stones. You can join any two piles together into one pile and you can divide a pile with an even number of stones into two piles of equal size. Can you ever achieve 105 piles, each with one stone? Explain.

13. Determine the value of $10^2 - 9^2 + 8^2 - 7^2 + 6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2$.