University of Waterloo
Faculty of Mathematics

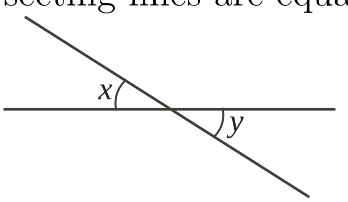
Centre for Education in
Mathematics and Computing

# Intermediate Math Circles
# October 29, 2008
# Geometry I

## Geometry

Geometry deals with angles and the relationships between them. There are a number of important theorems that illustrate these relationships.

The opposite angle theorem states that the opposite angles formed by two intersecting lines are equal. $\angle x = \angle y$



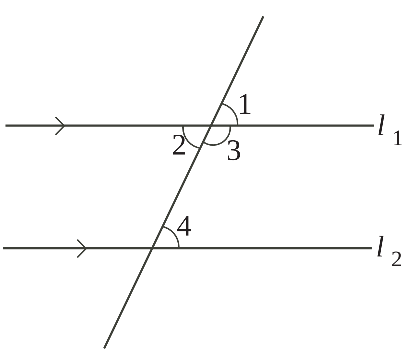

If two angles sum to 180°, then they are called supplementary angles.
If two angles sum to 90°, then they are called complementary angles.

The parallel line theorem is as follows. Suppose two lines $l_1$ and $l_2$ are parallel, and $m$ is a transverse line crossing them. Then

i) Corresponding angles are equal. $\angle 1 = \angle 4$ (F-pattern)

ii) Alternate angles are equal. $\angle 2 = \angle 4$ (Z-pattern)

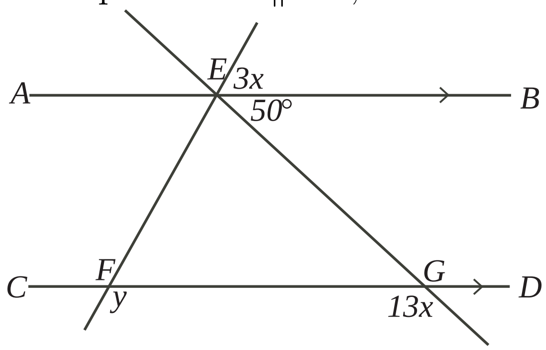
iii) Interior angles are supplementary. $\angle 3 + \angle 4 = 180°$ (C-pattern)




Also, if any of the three above conditions are true, then $l_1$ and $l_2$ are parallel.

Defintion:
An angle $x$ such that $0^\circ < x^\circ < 90^\circ$ is called *acute*, while an angle $y$ such that $90^\circ < y^\circ < 180^\circ$ is called *obtuse*.

Example: If $AB \| CD$, what are $x$ and $y$. Is each one acute or obtuse?



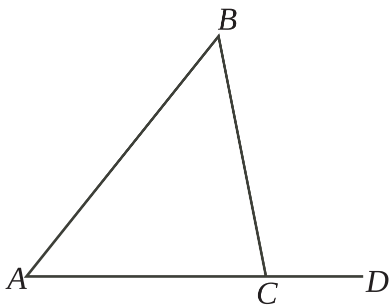

Solution:
$\angle BEG = \angle CGE = 50^\circ$
So, $13x + 50^\circ = 180^\circ$ or $13x = 130^\circ$ or $x = 10^\circ$.
Therefore, $x$ is acute.
$\angle EFG = 3x$
So $3x + y = 180^\circ$ or $30^\circ + y = 180^\circ$ or $y = 150^\circ$.
Therefore, $y$ is obtuse.

Polygons

In a triangle, the sum of the interior angles is $180^\circ$, and in a square the sum of the interior angles is $360^\circ$. The sum of the interior angles of a pentagon is $540^\circ$, and as you continue through the polygons, each with one more side than the last, the sum of their interior angles increases by $180^\circ$ each time. A general formula for the sum of the interior angles of a polygon with n sides (an $n-$gon) is $180^\circ(n-2)$.
A regular $n-$gon has each of its $n$ angles equal. Therefore, in a regular $n-$gon, the measure of each angle is $180^\circ(\frac{n-2}{n})$.

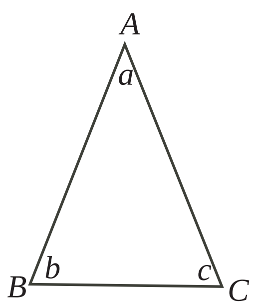
Triangles

The exterior angle theorem states that the exterior angle of one vertex of a triangle is equal to the sum of the interior angles at the other two vertices. For example, in the following diagram $\angle BCD = \angle CAB + \angle CBA$.




Definition:
*Equilateral* triangles are triangles with all three sides equal and all three angles equal.
*Scalene* triangles are triangles with all three sides different and all three angles different.
*Isosceles* triangles are triangles with two sides equal, and two angles equal.

The isosceles triangle theorem states that $AB = AC$ if and only if $b = c$.




The angle inequality theorem states that $\angle B > \angle C$ if and only if $b > c$.

The triangle inequality theorem states that in all triangles with side lengths $a, b$ and $c$, $a + b > c$, $a + c > b$, and $b + c > a$.

Example:
A triangle has sides of lengths $a, b$ and $c$. You know that $3 < a < 6$, $3 < b < 7$, and $6 < c < 10$, and $a, b, c$ are all integers. How many triangles can you form?

Solution:
$a$ must be 4 or 5, $b$ must be 4,5, or 6, and $c$ must be 7,8, or 9.

Let $a = 4$, $b = 4$, then $c = 7$ since $a + b > c$.
Let $a = 4$, $b = 5$, then $c = 7, 8$ since $a + b > c$.
Let $a = 4$, $b = 6$, then $c = 7, 8, 9$.

If we let $a = 5$ and $b = 4$, then we will have the same triangles as the first case when $a = 4$ and $b = 5$.
Let $a = 5$, $b = 5$, then $c = 7, 8, 9$.
Let $a = 5$, $b = 6$, then $c = 7, 8, 9$.
Therefore, we have 12 possible triangles.

Congruent Triangles

Two triangles are *congruent* if one of the following conditions is met:

i) SAS Side-Angle-Side: Two pairs of corresponding sides are equal, and the pair of angles between them are equal. Note that if the equal angles are not between the pairs of equal sides, this does not guarantee congruence.

ii) AAS Angle-Angle-Side: Two pairs of angles are equal, and any pair of two sides is equal.

iii) SSS Side-Side-Side: Three pairs of sides are equal.

iv) HS Hypotenuse-Side: In right triangles, the hypotenuses are equal and one pair of the other sides are equal.

Similar Triangles

Two triangles are *similar* if one of the following conditions is met:

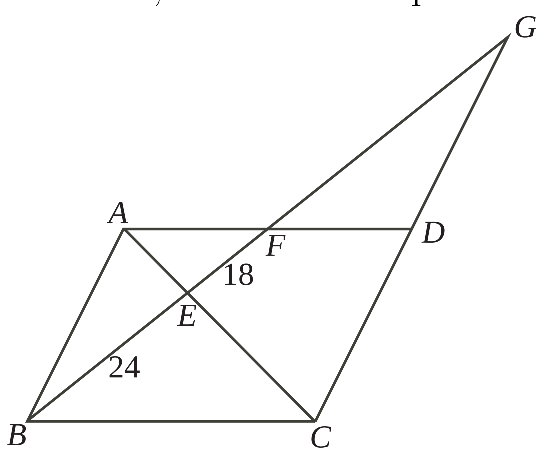
i) Two pairs of angles are equal.

ii) The ratios of pairs of sides are all equal.
$\frac{a}{d} = \frac{b}{e} = \frac{d}{f}$

Quadrilaterals

*Quadrilaterals* are polygons with four sides.
A *parallelogram* is a quadrilateral that has opposite sides parallel and opposite sides equal.
A *rhombus* is a quadrilateral that has opposite sides parallel and all sides equal.
A *rectangle* is a quadrilateral that has opposite sides equal and parallel, and has four right angles.
A *square* is a quadrilateral that has opposite sides parallel, four equal sides, and four right angles.

Example:
Find $FG$, if $ABCD$ is a parallelogram.



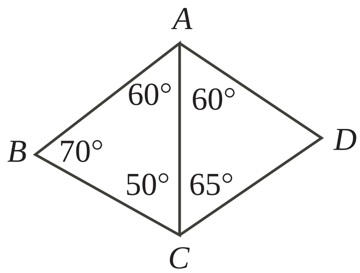

Solution:
Since $AF \| BC$ then $\angle EAF = \angle ECB$ and $\angle EFA = \angle EBC$.
Therefore $\triangle AEF \sim \triangle CEB$, so
$\frac{EC}{AE} = \frac{24}{18}$
Since $AB \| GC$ then $\angle ECG = \angle EAB$ and $\angle EGC = \angle EBA$.
Therefore $\triangle AEB \sim \triangle CEG$, so

$$\begin{aligned} \frac{EC}{AE} &= \frac{EG}{24} \\ \frac{24}{18} &= \frac{FG+18}{24} \\ 576 &= 18FG + 324 \\ 14 &= FG \end{aligned}$$

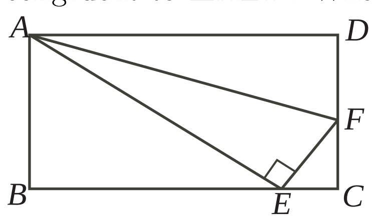
## Problem Set

1. How many scalene triangles exist which have all sides of integral length, and a perimeter less than 13?

2. In the diagram below, which side is longest: $AB, BC, AC, CD$, or $AD$?




3. In rectangle $ABCD$, $AD = 10$ and $AB = 8$. If point $F$ lies on side $CD$ such that $\triangle ADF$ is congruent to $\triangle AEF$. What is the length of $DF$?

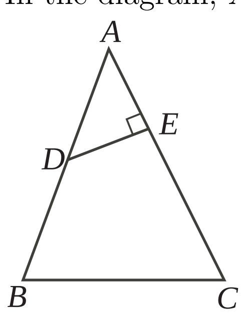



4. The legs of a right-angled triangle have lengths 5 and 10, while the hypotenuse of a similar triangle has length 15. What is the area of the larger triangle?

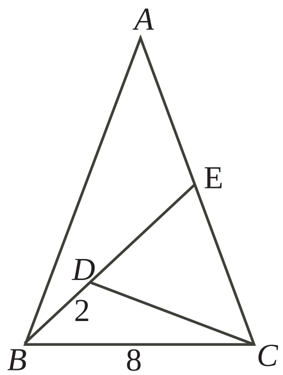
5. The sides of a triangle have lengths 10, 24, and 26. Find the shortest distance from the midpoint of the shortest side to the longest side.

6. What is the measure of the obtuse angle formed by the hands of a clock at 9:20?

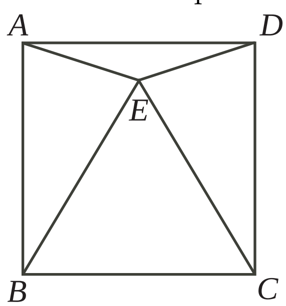
7. In the diagram, $AD = DB = 5$, $EC = 2AE = 8$, and $\angle AED = 90°$. Find $BC$.



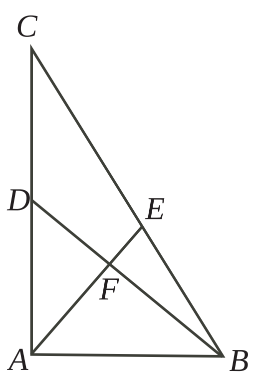
8. In the following diagram, $BD = 2$ and $BC = 8$. Also, $\angle ABC = \angle ACB = \angle CDE = \angle CED$. Find $AB$.




9. $ABCD$ is a square and $EBC$ is an equilateral triangle. What is $\angle AED$?

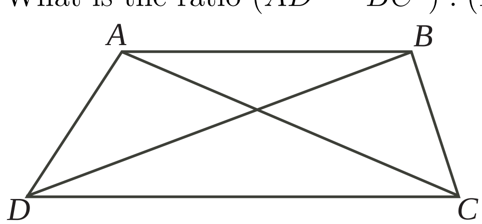



10. In $\triangle ABC$, medians $AE$ and $BD$ intersect at $F$. If $\angle BAC = \angle AFB = 90°$, and $AB = 12$, then what is the length of $BC$?

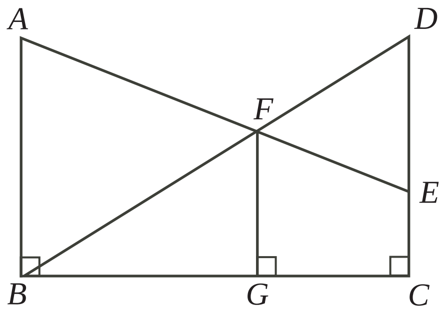



11. A rectangular sheet of paper $ABCD$ is three inches in width and four inches in length. The paper is folded so that the two diagonally opposite corners $A$ and $C$ coincide. Determine the length of the crease in the paper.

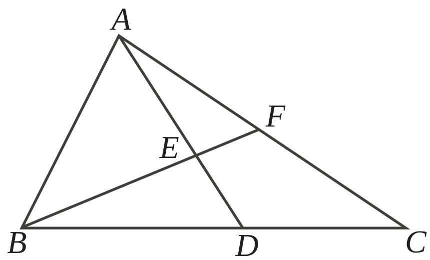
12. In the diagram $AB$ is parallel to $DC$, $AD > BC$, and the ratio $DC : AB = k$, where $k > 1$. What is the ratio $(AD^2 - BC^2) : (DB^2 - AC^2)$?



13. In the diagram, $DE : EC = 2 : 1$ and $AB = DC = 30$. Determine the length of $FG$.




14. In $\triangle ABC$, $D$ is the point dividing $BC$ in the ratio $5 : 3$ and $E$ is the point dividing $AD$ in the ratio $5 : 3$. $BE$ is extended to meet $AC$ at $F$. Determine the ratio $AF : FC$.




15. $ABC$ is an equilateral triangle, and $ABED$, $BCGF$, and $ACHI$ are squares. Prove that $FI = EH = DG$.