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Intermediate Math Circles
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Linear Equations I

Some definitions:

- A *linear equation* is an equation in which all the variables appear to the first power (exponent one).

- A *solution* to a linear equation is an *ordered n-tuple*, (pair, triple, quadruple,. . .), of real numbers which satisfy the linear equation.

- $\mathbb{R}^n$ is the set of all ordered n-tuples with real entries.

- A *system of linear equations*(linear system) is a set of one or more linear equations taken together.

- A *particular solution* to a system of linear equations is an *ordered n-tuple*, (pair, triple, quadruple,. . .), of real numbers which satisfies <u>all</u> of the linear equations in the system. To find a particular solution you have to give values to all but one of the variables and calculate the value of that variable.

- The *general solution* of a system of linear equations is the *set of all solutions* to the system of linear equations. These solutions will involve a *parameter*, or *parameters*, and so are sometimes called *parametric solutions*.

Exercise

Here are some examples of equations in $\mathbb{R}^2$. Determine a few particular solutions to the equations, and then determine a general solution:

1. $x = 3$
2. $x + y = 3$
3. $2x + 7y = 18$
4. $\frac{1}{2}x - 3y = -4$

Solutions

1. $(3,0),(3,2),(3,-3)$, and a general solution is $\{(3,y)|y \in \mathbb{R}\}$
2. $(3,0),(0,3),(10,-7),(-1,4)$, and general solutions are $\{(x,3-x)|x \in \mathbb{R}\}$ and $\{(3-y,y)|y \in \mathbb{R}\}$ In the general solutions, which are equivalent, the variables x and y are used as *parameters*.
3. $(9,0),(0,\frac{18}{7}),(2,2)$, and a general solution is $\{(x,\frac{18-2x}{7})|x \in \mathbb{R}\}$
4. $(-8,0),(0,\frac{4}{3}),(4,2)$, and a general solution is $\{(6y-8,y)|y \in \mathbb{R}\}$

Exercise

Here are some examples of equations in $\mathbb{R}^3$ and $\mathbb{R}^4$. Determine a few particular solutions to the equations, and then determine a general solution:

1. $\mathbb{R}^3$: $x + y + z = 3$
2. $\mathbb{R}^3$: $2x + 7y - 3z = 18$
3. $\mathbb{R}^3$: $y = 3$
4. $\mathbb{R}^3$: $\frac{1}{2}x - 3y + 17z = 0$
5. $\mathbb{R}^4$: $w + 2x + 3y + 4z = 42$

Solutions

1. $(3,0,0),(0,3,0),(0,0,3),(-1,4,0)$, and a general solution is $\{(x,y,3-x-y)|x,y \in \mathbb{R}\}$.
2. $(9,0,0),(0,\frac{18}{7},0),(0,0,-6),(-2,4,2)$, and a general solution is $\{(\frac{18-7y+3z}{2},y,z)|y,z \in \mathbb{R}\}$.
3. $(3,3,0),(0,3,0),(0,3,3),(-1,3,24)$, and a general solution is $\{(x,3,z)|x,z \in \mathbb{R}\}$.
4. $(0,0,0),(40,1,-1),(1,\frac{1}{2},\frac{1}{17})$, and a general solution is $\{(6y-34z,y,z)|x,z \in \mathbb{R}\}$.
5. $(3,3,3,6),(42,0,0,0),(0,21,0,0),(0,0,14,0)$, and a general solution is $\{(42-2x-3y-4z,x,y,z)|x,y,z \in \mathbb{R}\}$.

Solving Systems of Linear Equations

A system of linear equations can be thought of as a list of linear equations. In finding a solution to such a system there are three operations which are allowed. These are referred to as *elementary operations.*

Elementary Operations

a) Interchange the position of two equations in the list.

b) Multiply an equation by any non-zero constant.

c) Add a multiple of one equation to another equation.

These three operations can be performed on the systems as many times as necessary, using the updated systems, to reach a final solution, or to get to a point where substitutions can be made to solve the system.

Examples
Solve each of the following systems of linear equations:

1.
(1) $2x + y = 8$
(2) $x - y = 1$

Add equation (1) to equation (2) to obtain $3x = 9$, so $x = \frac{9}{3} = 3$.

Substitution into the equation (1) gives: $2(3) + y = 8$, so $y = 8 - 6 = 2$.

The solution to the system of linear equations is $(x, y) = (3, 2)$.

2.
(1) $2x + 3y = 6$
(2) $2x + 3y = -18$

Subtract equation (2) from equation (1) to obtain $0x + 0y = 24$.

Since no real numbers x and y will satisfy this equation, there are no solutions to the given system of linear equations.

3.
(1) $4x + 3y = 5$
(2) $8x + 6y = 10$

Add -2 times equation (1) to equation (2) to obtain $0x + 0y = 0$.

This tells us that every solution to equation (1) is also a solution to equation (2), so there are infinitely many solutions to the system of linear equations. A general solution is $\{(\frac{5-3y}{4}, y) | y \in \mathbb{R}\}$.

4.
$$(1) \quad 5x + y = 8$$
$$(2) \quad x - y = 1$$

Add equation (1) to equation (2) to obtain $6x = 9$, so $x = \frac{9}{6} = \frac{3}{2}$.

Substitution into the equation (2) gives: $\frac{3}{2} - y = 1$, so $y = \frac{3}{2} - 1 = \frac{1}{2}$.

The solution to the system of linear equations is $(x, y) = (\frac{3}{2}, \frac{1}{2})$.

5.
$$\begin{aligned} x + y + z &= 3 \\ x - y + 2z &= 13 \\ 3x + y - 3z &= -9 \end{aligned}$$

The only solution to the system of linear equations is $(x, y, z) = (-2, 3, 4)$.

6.
$$\begin{aligned} 2x - 7y + 4z &= -11 \\ x + 3y - 2z &= -1 \\ 3x - y + 5z &= 24 \end{aligned}$$

The only solution to the system of linear equations is $(x, y, z) = (-2, 5, 7)$.

7.
$$\begin{aligned} 6x + 2y - 3z &= 15 \\ 3 + 5y + 4z &= -8 \\ 4x + 5y + 13 &= 0 \end{aligned}$$

The only solution to the system of linear equations is $(x, y, z) = (15.5, -15, 16)$.

8.
$$\begin{aligned} x + y &= -1 \\ y + z &= 9 \\ x + z &= 4 \end{aligned}$$

The only solution to the system of linear equations is $(x, y, z) = (-3, 2, 7)$.

9. If a three digit number is decreased by 297 the result is the number with the digits reversed. Fifty times the sum of the digits is 32 less than the number. If the hundreds digit equals the sum of the other two digits, find the number.

   The number is 532.