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# Intermediate Math Circles
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# Linear Equations III

## Returning to Systems of Linear Equations

**Example** Solve the system of linear equations:

(1) $x + 2y - 3z = 0$

(2) $3x - 2y + z = 6$

(3) $5x + 2y - 5z = 6$

Solution:
Add equation (1) to equation (2) to get equation (4) $4x - 2z = 6$.
Subtract equation (1) from equation (3) to get equation (5) $4x - 2z = 6$.
Subtract equation (5) from equation (4) to get $0z = 0$, so we have eliminated one of the equations.
Now we have a system with 2 equations and 3 variables so the general solution will have one parameter, since # of equations + # of parameters = # of variables.
This esentially means that the system will have infinitely many solutions.
To find the general solution, we need to involve one parameter.
Using $x$ as our parameter, we need to find $z$ and $y$ in terms of $x$.
So, from equation (4), $z = 2x - 3$ and from equation (3), $y = \frac{5x-9}{2}$.
Thus, the general solution is $\{(x, y, z) = (x, \frac{5x-9}{2}, 2x - 3) | x \in \mathbb{R}\}$.

**Extra Problems:**

1. Solve the system of linear equations:

   (1) $3x + 4y + z = 6$

   (2) $2x - y + 8z = -7$

   (3) $2x + 5y - 4z = 11$

   Solution:
   Add 4 times equation (2) to equation (1) to get equation (4) $11x + 33z = -22$ or $x + 3z = -2$.
   Add 5 times equation (2) to equation (3) to get equation (5) $12x + 36z = -24$ or $x + 3z = -2$.
   Since these equations are the same, the system will have an infinite number of solutions and the general solution will have one parameter. Use $z$ as the parameter, $x = -2 - 3z$.
   Equation (2) gives $2(-2 - 3z) - y + 8z = -7$ or $y = 3 + 2z$.
   The general solution is $\{(x, y, z) = (-2 - 3z, 3 + 2z, z) | z \in \mathbb{R}\}$.

2. Solve the system of linear equations:

   $x + y - z = 2$

   $2x - y + z = 3$

   $5x - y + z = 8$

Solution:
Subtract 2 times equation (1) from equation (2) to get equation (4) $-3y + 3z = -1$.
Subtract 5 times equation (1) from equation (3) to get equation (5) $-6y + 6z = -2$ or $-3y + 3z = -1$.
Since these equations are the same, the system will have an infinite number of solutions and the general solution will have one parameter. Using $z$ as the parameter, $y = \frac{1}{3} + z$ and $x = \frac{5}{3}$.
The general solution is $\{(x, y, z) = (\frac{5}{3}, \frac{1}{3} + z, z) | z \in \mathbb{R}\}$.

3. Solve the following system of linear equations:

$$\begin{array}{lrcrcrcrcrcl}
(1) & x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0 \\
(2) & & & 2x_2 & + & x_3 & & & - & x_5 & = & 0 \\
(3) & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & = & 0 \\
(4) & x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0
\end{array}$$

Solution:
First, we can see that the system has more variables than equations. This means that if the equation has atleast 1 solution, it will have infinitely many solutions. The trivial solution where $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ is a solution to the system, so we know there will be infinitely many solutions.

Now, we can use the Gaussian elimination style to solve this system. First we will try to eliminate all but one of the '$x_1$'s using elementary operations.
Leave equation (1) since it will be the equation with an $x_1$ and it already has a coefficient of 1.
Leave equation (2) since there are no '$x_1$'s in that equation.
Subtract equation (1) from equation (3) to get equation (5) $-x_2 + x_3 - 2x_5 = 0$.
Subtract equation (1) from equation (4) to get equation (6) $-x_2 = 0$ or $x_2 = 0$.
Now, we are left with equations (1), (2), (5), and (6) where only (1) has an $x_1$.

$$\begin{array}{lrcrcrcrcrcl}
(1) & x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0 \\
(6) & & & x_2 & & & & & & & = & 0 \\
(5) & & & -x_2 & + & x_3 & & & - & 2x_5 & = & 0 \\
(2) & & & 2x_2 & + & x_3 & & & - & x_5 & = & 0
\end{array}$$

Leave equation (1) alone and eliminate all but one of the '$x_2$'s.
Leave equation (6) since it will be the equation with an $x_2$ and it already has a coefficient of 1.
Add equation (6) to equation (5) to get equation (7) $x_3 - 2x_5 = 0$.
Subtract 2 times equation (6) from equation (2) to get equation (8) $x_3 - x_5 = 0$.
Now, we are left with equations (1), (6), (7) and (8).

$$\begin{array}{lrcrcrcrcrcl}
(1) & x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0 \\
(6) & & & x_2 & & & & & & & = & 0 \\
(7) & & & & & x_3 & & & - & 2x_5 & = & 0 \\
(8) & & & & & x_3 & & & - & x_5 & = & 0
\end{array}$$

Leave equations (1) and (6) alone and eliminate all but one of the '$x_3$'s.
Subtract equation (7) from equation (8) to get equation (9) $x_5 = 0$.
Now, we are left with equations (1), (6), (7) and (9).

$$\begin{array}{lrcrcrcrcrcl}
(1) & x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0 \\
(6) & & & x_2 & & & & & & & = & 0 \\
(7) & & & & & x_3 & & & - & 2x_5 & = & 0 \\
(9) & & & & & & & & & x_5 & = & 0
\end{array}$$

The next step is to eliminate all but one '$x_4$'s but there is only one. We are done the Gaussian elimination.
From this, we can see that $x_5 = 0$ from equation (9) and if we plug that into equation (7) we get that $x_3 = 0$. We also know that $x_2 = 0$ from equation (6) and plugging all of this into equation (1) gives $x_1 + x_4 = 0$ or $x_1 = -x_4$.
Using $x_1$ as a parameter, the general solution is $\{(x_1, x_2, x_3, x_4, x_5) = (x_1, 0, 0, -x_1, 0) | x_1 \in \mathbb{R}\}$.

4. Find values of $a$, $b$, and $c$ such that the system of linear equations has

   a) exactly one solution
   b) an infinite number of solutions
   c) no solution.

$$\begin{array}{rcrcrcl} x & + & 5y & + & z & = & 0 \\ x & + & 6y & - & z & = & 0 \\ 2x & + & ay & + & bz & = & c \end{array}$$

5. For what value of the parameter $k$ does the linear system of equations

$$kx + y + z = k$$
$$x + ky + z = k$$
$$x + y + kz = k$$

   a) have no solution?
   b) have an infinite number of solutions?
   c) have precisely one solution?

6. For what value of the parameter $b$ does the linear system of equations

$$\begin{aligned} 4x + 3y + 3z &= -8 \\ 2x + y + z &= -4 \\ 3x - 2y + (b^2 - 6)z &= b - 4 \end{aligned}$$

   a) have exactly one solution?
   b) have no solutions?
   c) have more than one solution?

**Problem Set**

1. Solve

$$\begin{aligned} x(y + z - 2) &= 8 \\ y(x + z - 2) &= 8 \\ x^2 + y^2 &= 20 \end{aligned}$$

   where $x, y, z$ are integers.

2. Determine all solutions to the system of equations:

$$3x^2 - 2xy = 33$$
$$2x^2 - 3xy = 11$$

3. If $x$ and $y$ are real numbers, determine all solutions $(x, y)$ of the system of equations:

$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0$$

4. Determine all real values of $p$ and $r$ which satisfy the following system of equations:

$$p + pr + pr^2 = 26$$
$$p^2r + p^2r^2 + p^2r^3 = 156$$

## Answers

## Extra Problems

1. a) $\{a|a \in \mathbb{R}\}$, $\{b|b \neq -2a + 22, b \in \mathbb{R}\}$, $\{c|c \in \mathbb{R}\}$
   b) $\{a|a \in \mathbb{R}\}$, $\{b|b = -2a + 22\}, \{c|c = 0\}$
   c) $\{a|a \in \mathbb{R}\}$, $\{b|b = -2a + 22\}, \{c|c \neq 0, c \in \mathbb{R}\}$
2. a) $\{k|k = -2\}$
   b) $\{k|k = 1\}$
   c) $\{k|k \neq 1, k \neq -2, k \in \mathbb{R}\}$
3. a) $\{b|b \neq \pm 2, b \in \mathbb{R}\}$
   b) $\{b|b = 2\}$
   c) $\{b|b = -2\}$

## Problem Set

1. $\left(\sqrt{10}, \sqrt{10}, \dfrac{10 - \sqrt{10}}{5}\right), (4, 2, 2), (2, 4, 2), (-2, -4, 2), (-4, -2, 2)$
2. $\left(\sqrt{\dfrac{77}{5}}, \sqrt{\dfrac{99}{35}}\right), \left(-\sqrt{\dfrac{77}{5}}, -\sqrt{\dfrac{99}{35}}\right)$
3. $(-1, -9), (4 + 2\sqrt{2}, 8), (4 - 2\sqrt{2}, 8)$
4. $(2, 3), (18, \frac{1}{3})$