University of Waterloo
Faculty of Mathematics

Centre for Education in
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Intermediate Math Circles
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Linear Equations II

Solving Systems of Linear Equations

A system of linear equations can be thought of as a list of linear equations. In finding a solution to such a system there are three operations which are allowed. These are referred to as *elementary operations.*

Elementary Operations

a) Interchange the position of two equations in the list.

b) Multiply an equation by any non-zero constant.

c) Add a multiple of one equation to another equation.

These three operations can be performed on the systems as many times as necessary, using the updated systems, to reach a final solution, or to get to a point where substitutions can be made to solve the system.

Solutions for some of last week's questions:

5.
$$\begin{aligned} x + y + z &= 3 \\ x - y + 2z &= 13 \\ 3x + y - 3z &= -9 \end{aligned}$$

Solution

$$\begin{aligned} x + y + z &= 3 && (1) \\ x - y + 2z &= 13 && (2) \\ 3x + y - 3z &= -9 && (3) \end{aligned}$$

Add equation (2) to equation (1) to get equation (4) $2x + 3z = 16$.
Add equation (2) to equation (3) to get $4x - z = 4$ and rearrange this to get equation (5) $z = 4x - 4$.
Substitute equation (5) into equation (4) to get $x = 2$.
Substituting this into equation (5) gives $z = 4$, and substituting these two values back into

equation (1) gives $y = -3$.

The solution is $(2, -3, 4)$.

9. If a three digit number is decreased by 297 the result is the number with the digits reversed. Fifty times the sum of the digits is 32 less than the number. If the hundreds digit equals the sum of the other two digits, find the number.

   Solution

   Let x represent the hundred's digit of the number, y represent the ten's digit, and z the unit's digit.

   From the statement: "If a three digit number is decreased by 297 the result is the number with the digits reversed.", the following equation is obtained,

   $$100x + 10y + z - 297 = 100z + 10y + x$$

   or

   $$99x - 99z = 297, \text{ which is the same as } x - z = 3 \ (1)$$

   From the statement: "Fifty times the sum of the digits is 32 less than the number.", the following equation is obtained,

   $$50(x + y + z) = 100x + 10y + z - 32$$

   or

   $$50x - 40y - 49z = 32 \ (2)$$

   From the statement: "the hundreds digit equals the sum of the other two digits," the following equation is obtained,

   $$x = y + z$$

   or

   $$x - z = y \ (3)$$

   Putting the first and third equations together, we obtain $y = 3$.

   Adding -49 times equation (1) to equation (2), with $y = 3$ substituted, gives $x = 5$.

   Substitution of $x = 5$ and $y = 3$ into equation (3) gives $z = 2$.
   The solution to the system of equations is $(x, y, z) = (5, 3, 2)$, and the number described in the question is 532.

Linear Diophantine Equations

Linear Diophantine equations are simply linear equations which only use integers. This means they have integer coefficients and when we solve them, we are only looking for integer solutions.

Division algorithm: Express 54 in the form $54 = 11 \times q + r$ with $0 \leq r < 11$.

This can be done in only one way: $54 = 11 \times 4 + 10$, that is $q = 4$ and $r = 10$.

The division algorithm states that given two integers a and b, there exist unique q and r such that $a = qb + r$ with $0 \leq r < |b|$.

The Greatest Common Divisor, GCD, of two numbers is the largest integer that divides evenly into both numbers.
The Euclidean Algorithm uses the division algorithm repeatedly, and gives the Greatest Common Divisor, GCD, of the numbers you start with.

Example

a) Find the GCD of 506 and 391, or $GCD(506, 391)$.

$$\begin{aligned} 506 &= 391 \times 1 + 115 \\ 391 &= 115 \times 3 + 46 \\ 115 &= 46 \times 2 + 23 \\ 46 &= 23 \times 2 + 0 \end{aligned}$$

The last non-zero remainder, r, is the greatest common divisor of the numbers we start with, so $GCD(506, 391) = 23$.

b) Solve the linear Diophantine equation $506x + 391y = 23$.

This means we want solutions to the equation where both x and y are integers.
We use the Euclidean Algorithm's steps in reverse.
The second last line tells us that $23 = 115 - 2(46)$.
From the second line we have, $46 = 391 - 3(115)$ and substitution gives:

$$\begin{aligned} 23 &= 115 - 2(46) \\ 23 &= 115 - 2(391 - 3(115)) \\ 23 &= -2(391) + 7(115) \end{aligned}$$

From the first line we have $115 = 506 - 1(391)$, and substituting this gives:

$$\begin{aligned} 23 &= -2(391) + 7(115) \\ 23 &= -2(391) + 7(506 - 1(391)) \\ 23 &= 7(506) - 9(391) \end{aligned}$$

Thus $(x, y) = (7, -9)$ is an integer solution to the equation.

c) Solve the linear Diophantine equation $506x + 391y = 92$.

We notice that $92 = 4 \times 23$ and since $7(506) - 9(391) = 23$, then
$4(7(506) - 9(391)) = 4(23)$
$28(506) - 36(391) = 92$
So a solution is $(28, -36)$, which is simply four times the solution to $506x + 391y = 23$.

Generally, the linear diophantine equation

$$ax + by = c$$

has a solution if and only if the $GCD(a, b)$ divides c,

and

if $GCD(a, b) = d \neq 0$, and $x = x_0, y = y_0$, is a particular solution, then the complete integer solution is

$$x = x_0 + n\frac{b}{d}, \qquad y = y_0 - n\frac{a}{d} \qquad \text{, for all integers } n.$$

Example (cont.)

d) Using this theorem we can give the complete solution to the Diophantine equation $506x+391y = 23$.

The complete solution is:

$$x = 7 + n\left(\frac{391}{23}\right) = 7 + 17n, y = -9 - n\left(\frac{506}{23}\right) = -9 - 22n, \text{ for } n \text{ an integer.}$$

Problem Set

1. Find the GCD of each of the following pairs of integers:
   a) 5280 and 3600
   b) 484 and 451
   c) 616 and 427
   d) 1137 and -419
2. Find all solutions to each Diophantine equation:
   a) $7x + 9y = 1$
   b) $212x + 37y = 1$
   c) $243x + 405y = 123$
   d) $169x - 65y = 91$
3. Find all non-negative solutions to each Diophantine equation:
   a) $14x + 9y = 1000$
   b) $12x + 57y = 423$

Answers

1. (a) 240
   (b) 11
   (c) 7
   (d) 1
2. (a) $x = 4 + 9n, y = -3 - 7n$
   (b) $x = 11 + 37n, y = -63 - 212n$
   (c) no solutions
   (d) $x = 14 - 5n, y = 35 - 13n$
3. (a) $(65, 10), (56, 24), (47, 38), (38, 52), (29, 66), (20, 80), (11, 94), (2, 108)$
   (b) $(21, 3), (2, 7)$