University of Waterloo
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Intermediate Math Circles
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Counting III

Last time, we looked at combinations and saw that we still need to use the product and sum rule to solve many of the problems. Today, we start by looking at how to count the number of permutations when the objects are not all distinct. We will then look at permutations on combinations when repetitions are allowed.

Example 1:
Calculate the number of possible permutations on the set $\{A, A, B, B, B\}$.

Solution:
Lets first list all the possibilities and then try to figure out how we could have calculated the number. We get

$$AABBB, ABABB, ABBAB, ABBBA, BAABB,$$
$$BABAB, BABBA, BBAAB, BBABA, BBBAA$$

Observe that instead of thinking about how to fill each place, we were more thinking about how to arrange the A's. How many ways were there to do this? There were 5 possible spots and 2 choices for A so $\binom{5}{2} = 10$. This is all the possibilities since the B's must fill in the remaining spots.

Of course, we could have instead counted the number of ways we could have placed the 3 Bs into the 5 spots and we would have got $\binom{5}{3} = 10$.

Example 2:
Calculate the number of possible permutations on a set with m A's and n B's.

Solution:
We have $m + n$ spots and so we have $\binom{m+n}{m}$ ways of placing the A's, which is thus the total number of permutations.

Example 3:
Calculate the number of permutations on a set with 3 A's, 6 B's and 3 C's.

Solution:
We now see that we have 12 possible positions. We will break this problem down into 2 easy problems. First, let us just think of the problem as considering that the B's and C's are the same. Then, from

example 2 we know that there are $\binom{12}{3}$ permutations. But, since the B's and C's are not the same, after we place those A's we will have 9 remaining positions to fill in the 6 B's and 3 C's. But, from example 2, we know there are $\binom{9}{6}$ ways of doing this. Then, we are down to 3 empty positions and so there is only one way to fill in the C's. Thus, by the product rule we have the total number of permutations on the set is

$$\binom{12}{3} \times \binom{9}{6} \times 1 = \frac{12!}{3!9!} \times \frac{9!}{6!3!} = \frac{12!}{3!6!3!}.$$

Exercises

1. How many permutations are there on a set with 2 A's, 4 B's, 3 C's, 2 D's.
2. How many permutations are there on the letters of "CALCULATE".
3. How many permutations are there on the letters of "LETTERS" that start with an R?

Answers:

1. $\binom{11}{2} \times \binom{9}{4} \times \binom{5}{3} \times 1 = \frac{11!}{2!9!} \times \frac{9!}{4!5!} \times \frac{5!}{3!}2! = \frac{11!}{2!4!3!2!}$
2. $\binom{9}{2} \times \binom{7}{2} \times \binom{5}{2} \times 3! = \frac{9!}{7!2!} \times \frac{7!}{2!5!} \times \frac{5!}{2!3!} = \frac{9!}{2!2!2!}$.
3. $\binom{6}{2} \times \binom{4}{2} \times 2! = \frac{6!}{2!2!}$.

Permutations with Repetition

We now look at permutations with repetition allowed. That is how many ways can we choose r objects out of n objects when we are allowed to select any of the n objects more than once.

Example 4:
Anna Lize gets to pick a 4-digit combination for her combination lock. How many different combinations are possible?

Solution:
This is a permutation with repetition problem since she needs to pick 4 digits from the set $\{0, 1, \ldots, 9\}$. However, we also see that this is just a simple product rule problem. In each case she has 10 choices for each digit, so the total number of combinations is 10^4.

In particular, it is easy to see that permutations with repetitions are just product rule problems. There are always n^r ways of choosing r objects out of n objects when repetition is allowed. Thus, we move onto the more interesting problem of combinations with repetition.

Combinations with Repetition

Example 5:

A pop machine offers 6 kinds of pop. Polly Knowmeal wants to purchase 4 cans. How many different purchases can she make?

Solution:

Lets call the types of pop A,B,C,D,E,F. Observe that we are not interested in the order that she purchases the pop, but instead how many of each type she purchases. Let us list some possibilities. We just list them by type since order does not matter.

$$AABC \quad ABDE \quad DEFF \quad FFFF.$$

Observe that since we are just counting types, I could instead represent these by

$$XX|X|X||| \quad X|X||X|X| \quad |||X|X|XX \quad |||||XXXX.$$

We observe that we can instead think of this problem as trying to arrange the 4 X's among the 5 $|$'s. Thus, we are putting X's and $|$'s into 9 spots. From last time, we know that the number of ways of doing this is $\binom{9}{4}$.

Example 6:

In how many ways can you put 10 pennies into 5 jars?

Solution:

If we represent the jars by $|$, then we see that we are just trying to arrange how to put the 10 pennies among the 4 $|$'s. i.e. one arrangement is

$$pp|pp|ppp|p|pp$$

Thus, as in the last example, we see that we are trying to arrange the 10 p's into 14 spaces. Hence the total number of ways of doing this is $\binom{14}{10}$.

In general, we see that the number of ways to select r objects out of n where repetition is allowed and order does not matter is

$$\binom{r+n-1}{r}$$

since the number of $|$'s is $n-1$.

Example 7:

In how many ways can 5 calculators and 7 sledgehammers be distributed among 4 students so that each student receives at least one sledgehammer?

Solution:

If each student gets a sledgehammer then there are 3 more sledgehammers to distribute among the 4 students which can be done in $\binom{3+4-1}{3} = \binom{6}{3}$ ways. Then, we have $\binom{5+4-1}{5} = \binom{8}{5}$ ways of distributing the 5 calculators among the 4 students. Hence, by the product rule we get $\binom{6}{3} \times \binom{8}{5}$ ways.

Exercises

1. At a cafeteria, a student is allowed to pick 4 items from the following list

$$\{ \text{ pop, juice, milk, burger, soup, banana, orange, apple pie} \},$$

   and they are allowed to pick the same item more than once. How many different meals can they pick?

2. Anna Lize has 10 loonies and wants to distribute them among her 3 friends. In how many ways can she do this if:

   a.) it is okay if one or more of her friends get nothing.

   b.) each friends gets at least 1 loonie.

   c.) Her best friend, Ann Alysis, gets at least 5 loonies and the other 2 friends gets at least 1 loonie.

3. Determine how many all integer solutions there are to the equation $x_1 + x_2 + x_3 + x_4 = 7$, if none of the x_i are allowed to be negative?

Answers:

1. $\binom{11}{4}$
2. a) $\binom{12}{3}$, b) $\binom{9}{3}$, c) $\binom{5}{3}$ or $\binom{4}{3} + \binom{3}{2} + \binom{2}{1} + 1$.
3. $\binom{10}{7}$.

Observe from the last exercise that the following are all equivalent:

1. The number of integer solutions of the equation $x_1 + x_2 + \cdots + x_n = r$, $x_i \geq 0$.
2. The number of selections of r objects from n objects where repetition is allowed.
3. The number of ways r objects can be distributed among n distinct containers.

Example 8:
Determine all the ways in which we can write 4 as a sum of positive integers.

Solution:
We will list them as we count them.

As a sum of **one positive integer**: only one way ... i.e 4

As a sum of **two positive integers**: This is asking the number of solutions to $x_1 + x_2 = 4$, $x_i > 0$. Since, $x_i > 0$, that means x_1 and x_2 are at least one. So we can look at the problem as $(y_1 + 1) + (y_2 + 1) = 4$ so $y_1 + y_2 = 2$. So the number of ways is $\binom{3}{2} = 3$.. .i.e. 1+3, 3+1, 2+2

As a sum of **three positive integers**: Using the same argument we get $y_1 + y_2 + y_3 = 1$, $y_i \geq 0$... which gives $\binom{3}{1} = 3$ ways ... i.e 1+1+2, 1+2+1, 1+1+2.

As a sum of **four positive integers**: one way .... 1+1+1+1.... i.e solving $y_1+y_2+y_3+y_4=0$, $y_i \geq 0$.

So, the total number of ways is

$$\binom{3}{3}+\binom{3}{2}+\binom{3}{1}+\binom{3}{0}=2^3.$$

Problem Set

1. How many permutations can be formed from the letters of "MNEMONIC".
2. In how many ways can Mrs I. Sosceles distribute 10 identical candy bars among her 5 children if each child gets at least one candy bar?
3. Determine the number of integer solutions of $x_1+x_2+x_3+x_4=32$, where

   a.) $x_i \geq 0$, $1 \leq i \leq 4$.

   b.) $x_i > 0$, $1 \leq i \leq 4$.
4. In how many ways can you distribute 8 identical balls into 4 distinct containers so that the fourth container has an odd number of balls in it?
5. How many permutations of the letters of "MISSISSAUGA" do not start with S?
6. Determine how many ways in which we can write 7 as a sum of positive integers.

Answers to last week's Problem Set

1. $\binom{50}{30}$; 2. a) $\binom{8}{2}\binom{5}{2}$, b) $\binom{8}{1}\binom{5}{1}+\binom{8}{2}\binom{5}{2}+\binom{8}{3}\binom{5}{4}+\binom{8}{4}\binom{5}{4}+\binom{8}{5}\binom{5}{5}$

3. a) $\binom{18}{6}$, b) $\binom{8}{3}\binom{11}{3}$, c) $6\times\binom{13}{5}$; 4. a) $\binom{52}{5}$, b) $13\times\binom{4}{2}\times\binom{12}{3}\times 4^3$, c) 13×48

5. a) $\binom{10}{3}\times\binom{10}{4}\times 7!$, b) $\binom{9}{2}\times\binom{11}{5}\times 7!$, c) $\binom{20}{7}$; 6. a) 1, b) $\frac{55}{14}$, c) $\frac{n-2}{3}$.

Answers to this week's Problem Set

1. a) $\frac{8!}{2!2!}$; 2. $\binom{9}{5}$; 3. a) $\binom{35}{32}$, b) $\binom{31}{28}$

4. $\binom{9}{7}+\binom{7}{5}+\binom{5}{3}+\binom{3}{1}$; 5. $\frac{11!}{4!2!2!}-\frac{10!}{3!2!2!}$. 6. 2^6.