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Intermediate Math Circles
November 4, 2009
Counting II

Last time, after looking at the product rule and sum rule, we looked at counting permutations. In particular, at how many ways in which we could choose k object from n distinct objects where the order we picked them mattered. Today, we will look at **combinations**: the number of ways in which we can choose k objects from n distinct objects, when order does not matter.

Example 1:
A math student is given a list of 5 math problems and is asked to do any 3 of the problems. How many different choices can the student make?

Solution:
Let us name the problems 1, 2, 3, 4, and 5. Then, the possible choices are:

$$\{1,2,3\} \quad \{1,2,4\} \quad \{2,4,5\} \quad \{1,2,5\} \quad \{1,3,4\}$$

$$\{1,3,5\} \quad \{2,3,4\} \quad \{2,3,5\} \quad \{3,4,5\} \quad \{1,4,5\}$$

Observe that since it does not matter which order the student does the problems, the choice $\{1,2,3\}$ would be the same as the choice $\{3,1,2\}$ of $\{2,3,1\}$, etc.

We want to figure out how to count these mathematically. One way we can think about this is we want to count all the permutations of the n objects taken k at a time, and then remove all the permutations which are the same.

In the example above, what are the total number of permutations? $5 \times 4 \times 3 = 60$.

For any particular permutation, how many permutations are equal to it? If we consider the permutation $\{(1,2,3)\}$, then any permutation containing just these 3 numbers are the same.... how many of these are there? This is just all the permutations on 3 objects so $3! = 6$

Hence, we have 60 total permutations, but we can organize these into groups of 6, which are all equivalent if we are saying order does not matter. Hence, the total number of possible choices is 60 objects divided by 6 groups = 10, which matches what we did above.

Thus, the number of ways in which we can choose k objects from n distinct objects when order does not matter is

$$\frac{n \times (n-1) \times \cdots \times (n-(k-1))}{k!}$$

The formula does look very nice in this form, so we will try to make it look prettier. Observe that $5 \times 4 \times 3 = \frac{5\times4\times3\times2\times1}{2\times1} = \frac{5!}{2!}$, in particular

$$n \times (n-1) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}$$

Thus, we can write the number of permutations on n objects taken k at a time as $\frac{n!}{(n-k)!}$. So, the number of ways in which we can choose k objects from n distinct objects is

$$\frac{n!}{(n-k)!k!}.$$

We denote this with the symbol $\binom{n}{k}$ and say "n choose k".

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Example 2:
At a cafeteria, a student is allowed to pick 4 items from the following list:

{ pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, apple pie }.

a.) How many different choices does the student have?

Solution:

$$\binom{10}{4} = \frac{10!}{6!4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 10 \times 3 \times 7 = 210.$$

b) How many different choices does the student have, if they don't like apple pie?

Solution:

$$\binom{9}{4} = \frac{9!}{6!4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 3 \times 7 \times 6 = 126.$$

c) How many different choice does the student have, if they must pick one, and only one drink?

Solution:
The student has 4 choices to which drink they could pick. Then, the student can pick any 3 of the remaining 6 items so they have $\binom{6}{3}$ ways of picking those. Hence, by the product rule, the student has

$$4 \times \binom{6}{3} = 4 \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 4 \times 20 = 80, \text{ choices}$$

Exercises

1. Evaluate the following:

   a.) $\binom{7}{3}$ b.) $\binom{10}{2}$ c.) $\dfrac{\binom{6}{4}}{\binom{6}{2}}$

2. For a quest, the knight, Sir Cumference, needs to pick 4 out of his 10 fellow knights.

   a.) In how many ways can he do this?

   b.) In how many ways can he do this, if he must pick one particular knight, Sir Kull.

   c.) In how many ways can he do this, if he can't pick Sir Kull?

3. A student taking a math test and is told to answer any 7 of the 10 questions. In how many ways can the student do this?

4. A high school of 520 boys and 480 girls must select 5 students to represent them at a competition.

   a.) In how many ways can they do this?

   b.) In how many ways can they do this if boys are not allowed to be selected?

   c.) In how many ways can they do this if the group must have 3 boys and 2 girls?

   d.) In how many ways can they do this if the group must have more boys than girls?

Answers:

1. a) 35, b) 45, c) 1
2. a) $\binom{10}{4} = 210$, b) $\binom{9}{3} = 84$, c) $\binom{9}{4} = 126$
3. $\binom{10}{7}$
4. a) $\binom{1000}{5}$, b) $\binom{480}{5}$, c) $\binom{520}{3} \times \binom{480}{2}$, d) $\binom{520}{5} + \binom{520}{4} \times \binom{480}{1} + \binom{580}{3} \times \binom{480}{2}$

Pascal's Triangle

If one looks at values of $\binom{n}{k}$ as we change n and k, one quickly notices a pattern. Let us do this and find the pattern.

We have $\binom{0}{0}$... hmmm, this is the number of ways we can pick 0 objects from 0 objects.... we are not really doing much then, so we will say there is just 1 way of doing this. So $\binom{0}{0} = 1$.

What is $\binom{1}{0}$? How many ways can we pick 0 objects from 1 object? 1 way... we don't pick any objects. So $\binom{1}{0} = 1$. Notice then that we have $\frac{1!}{0!1!} = 1$... so we define $0! = 1$.

What is $\binom{1}{1}$? How many ways can we pick 1 object from 1 object? 1 way... we pick the object. So $\binom{1}{1} = 1$.

We the calculate that $\binom{2}{0} = \frac{2!}{2!0!} = 1$, $\binom{2}{1} = \frac{2!}{1!1!} = 2$, $\binom{2}{2} = \frac{2!}{0!2!} = 1$.

Lets start making these into a table in the form of a triangle

$$\begin{array}{ccccc} & & \binom{0}{0} & & \\ & \binom{1}{0} & & \binom{1}{1} & \\ \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} \end{array}$$

Substituting in the values gives

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Exercise: Fill in the next 2 rows. Try to find the pattern in the triangle and use it to fill in 2 more rows.

Answer: Observe that the ends of each row are always 1 and the middle numbers are just the sum of the two numbers above it. This gives the triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

which is now called Pascal's triangle.

Pascal's triangle has many interesting properties and uses. One immediate use is that it suggests some properties of $\binom{n}{k}$. In particular, notice that $\binom{n}{k} = \binom{n}{n-k}$.

But, Pascal's triangle is not a proof of this fact... so let us prove it.

Proposition: For any non-negative integers n and k with $n \geq k$ we have $\binom{n}{k} = \binom{n}{n-k}$.

Proof: We have $\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$. □

The triangle also suggests another property (which is much harder to prove), namely that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Compare this now with Exercise 1 above. Observe that we in fact had

$$\binom{10}{4} = \binom{9}{3} + \binom{9}{4}.$$

Which follows from the sum rule.... the number of ways of choosing the knights is the number of ways of choosing a group with Sir Kull plus the number of ways of choosing a group without Sir Kull.

Exercises

1. **a.)** Mr E. Lipps has in front of him a circle, a square, and a rectangle. In how many ways can he select some of the shapes? (he may pick any number of shapes)

   b.) If he has 4 shapes instead of three, in how many ways can he select some of the shapes? What if he had n shapes?

2. How many games are held in a round-robin singles tennis tournament involving n players?

3. Determine $\frac{\binom{n}{k}}{\binom{n}{k-1}}$.

Answers:

1. a) $7 = 2^3 - 1$, b) $15 = 2^4 - 1$, $2^n - 1$.
2. $\binom{n}{2} = \frac{n(n-1)}{2}$.
3. $\frac{n-k+1}{k}$.

Problem Set

1. If 50 different students try out for a team of 30 players, in how many different ways can the coach choose the team?

2. How many groups can be formed from 8 men and 5 women if

   a.) the group must have 2 women and 2 men?

   b.) the group can be any size, but must have an equal number of men and women?

3. How many subsets of size 7 of the set $\{1, 2, \ldots, 20\}$ have

   a.) 19 as the largest element?

   b.) 9 as the middle element?

   c.) the difference between the largest and smallest elements equal to 14?

4. With a standard deck of 52 cards, a subset of 5 cards is called a hand.

   a.) How many hands are there?

   b.) How many hands contain one pair? (2 of a kind and 3 different cards)

   c.) How many hands have 4 of a kind?

5. How many **permutations** of the numbers $1, 2, \ldots, 20$ taken 7 at a time

   a.) contain 3 odd and 4 even numbers?

   b.) contain 2 single-digit numbers and 5 two-digit numbers?

   c.) have the 7 numbers arranged in increasing order?

6. Evaluate the following **without** the use of a calculator.

   a) $\frac{\binom{7}{3}}{\binom{7}{4}}$ b) $\frac{\binom{12}{8}}{\binom{9}{4}}$ c) $\frac{\binom{n}{3}}{\binom{n}{2}}$, $n \geq 3$.

Answers to last week's Problem Set

1. a) 8, b) 15;
2. 500
3. $26^4 \times 10^3$
4. 9^3
5. 720
6. a) $3 \times 5!$, b) $9 \times 4!$, c) $6 \times 4!$
7. $2 \times 4 \times 6 \times 5 \times 4$, b) $2 \times 3 \times 6 \times 5 \times 4$