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Intermediate Math Circles
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Rates

A rate is a ratio of a quantity to an amount of time. The quantity could be a distance, a volume, a number of items.

For example 100 $\frac{km}{h}$, litres per minute, and 15 cars per day are all rates.

All rate problems depend on one basic equation:

$$RATE = \frac{QUANTITY}{TIME}$$

Given any two of these variables, you can solve for the third.

Example:
A hose expels water at a flow rate of 15 litres per minute for 5 minutes. How much water does it put out in this time?

Sol$^{\underline{\mathbf{n}}}$: Let the amount of water equal w.

Then $15\frac{L}{min} = \dfrac{w}{5\ min}$, so $w = (15\frac{L}{min})(5min) = 75L$

Therefore the hose puts out 75L of water.

Often times in rate problems there are two objects moving at different rates and we may want to figure out the rate at which they are moving apart.

Example:
Norma leaves her house travelling east at 80 $\frac{km}{h}$ at the same time that Kelly leaves the house travelling west at 60 $\frac{km}{h}$. How far apart are they after one hour? After two hours?
Sol$^{\underline{n}}$: Let n be the distance Norma travels in an hour and k be the distance Kelly travels in an hour.

$$\begin{aligned} n &= (80\tfrac{km}{h})(1h) \\ &= 80\ km \end{aligned} \qquad \begin{aligned} n &= (60\tfrac{km}{h})(1h) \\ &= 60\ km \end{aligned}$$

Therefore, after one hour they are $80\ km + 60\ km = 140\ km$ apart.

After two hours they will be $(2)(80\ km) + (2)(60\ km) = 280\ km$ apart.

This illustrates the basic concept that when we have Object A moving at rate a from and Object B moving at rate b in opposite directions, their rate of separation is $a + b$. If Object A and Object B moved in the same direction, their rate of separation would be $|a - b|$ where the bars denote the absolute value (the positive value).

Example:
A horse gallops away from a barn at 15 $\frac{m}{s}$ at which time a human immediately starts running after it at 8 $\frac{m}{s}$. How far ahead will the horse be after 5 seconds? After how long will the horse be 100 m ahead?

Sol$^{\underline{n}}$: The rate of separation of the horse and the human is $|8\frac{m}{s} - 15\frac{m}{s}| = |-7\frac{m}{s}| = 7\frac{m}{s}$
Therefore, after 5 seconds the horse will be$(7\frac{m}{s})(5s) = 35m$ ahead.
Let t be the time at which the horse will be 100 m ahead.

$$\begin{aligned} 7\tfrac{m}{s} &= \frac{100m}{t} \qquad \text{so if we rearrange this equation} \\ t &= \frac{100m}{7\frac{m}{s}} \\ &= 14.3s \end{aligned}$$

Therefore, after approximately 14.3 s the horse will be 100 m ahead.

Another type of problem involves the rate at which several people can complete an activity when combined compared to when alone.

Example:
A long driveway needs to be shoveled. Al knows he can finish shoveling it in 3 hours, Bob can finish it in 4 hours, Carl can finish it in 6 hours and Doug can finish it in 8 hours. If they all work together, how long will it take them to finish shoveling the driveway?

Sol$^{\underline{n}}$: In one hour Al shovels $\frac{1}{3}$ of the driveway, Bob shovels $\frac{1}{4}$, Carl shovels $\frac{1}{6}$ and Doug shovels $\frac{1}{8}$.

All together, in one hour, they shovel $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}=\frac{8}{24}+\frac{6}{24}+\frac{4}{24}+\frac{3}{24}=\frac{21}{24}=\frac{7}{8}$ of the driveway.

Let t be the amount of time it takes them to shovel the entire driveway.

$$\frac{\frac{7}{8}}{1h}=\frac{1}{t}$$

Therefore $t=\frac{8}{7}$hours $\approx$ 68.6minutes

Example:
Jack and John are supposed to blow up balloons for a party. Jack knows that it will take him 8 hours to blow up the balloons by himself, and John knows that it will take him 6 hours to blow up the balloons by himself. They plan to work together to get it done faster, but John shows up late to help and Jack has already blown up half of the balloons. What was the total amount of time it took to blow up all the balloons.

Sol$^{\underline{n}}$:
Since Jack takes 8 hours to blow up all the balloons, it must have taken him 4 hours to blow up half of them.

Jack can blow up $\frac{1}{8}$ of the balloons in an hour, and John can blow up $\frac{1}{6}$ of the balloons in an hour. Together, they can blow up $\frac{1}{8}+\frac{1}{6}=\frac{3}{24}+\frac{4}{24}=\frac{7}{24}$ of the balloons in one hour.

Let t be the time it takes them to blow up half of the balloons.

$$\frac{\frac{7}{24}}{1h}=\frac{\frac{1}{2}}{t}$$

$$(\tfrac{7}{24})(t)=(\tfrac{1}{2})(1h)$$

$$t=\frac{12}{7}$$

Therefore, the total time it took is, $4h+\frac{12}{7}h=5\frac{5}{7}h\approx 5\,h\;43\,min.$

Problem Set

1. Alf is in Alliston and Barb is in Beachtown, which is 360 km away from Alliston. At 9:00 a.m. Alf starts driving towards Beachtown at 80 $\frac{km}{h}$ and Barb starts driving towards Alliston at 40 $\frac{km}{h}$. At what time do they pass each other, and how far away from Alliston are they at this point?

2. Patrick leaves Whitby at 5:00 p.m. and is supposed to meet a friend in Waterloo, 150 km away, at 7:00 p.m. He gets stuck in traffic for the first half hour and only manages a speed of 30 $\frac{km}{h}$ during that time. How fast must he drive the rest of the way in order to make it to Waterloo on time?

3. Two candles of equal length are lit at noon. One candle takes 9 *hours* to completely burn, and the other takes 6 *hours* to completely burn. At what time will the slower burning candle be exactly twice as long as the faster burning one?

4. Two runners, Cara and Diane, run a 400 m race, and Diane wins. They run the race again at the same speeds as they ran before, but this time Diane starts from 50 m behind the start line, and they finish at the same time. How far was Cara from the finish line when Diane won the first race?

5. Emma and Julie start at the same point on a 500m circular track and run in opposite directions, at constant but different speeds. They pass each other, running in opposite directions, after 54 *seconds*. If Emma, who is faster, completes a lap of the track in 90 *seconds*, how long does it take Julie to run a lap?

6. Two horsemen spot each other from 400 m apart, and start riding towards each other, one at 2 $\frac{m}{s}$ and the other at 3 $\frac{m}{s}$. A fly starts at one horse and, flying at 5 $\frac{m}{s}$, flies to the other horse, turns around and immediately flies back. If the fly continues flying back and forth until the horses meet, what total distance does the fly cover?

7. Janices backyard pool springs a leak, and begins draining at a rate of 2 litres per minute. After an hour Janice notices the water level is lower than usual, and begins filling the pool with a hose that outputs water at 10 litres per minute. If the pool continues to leak while she is refilling it, after how long will it be full again?

8. An airplane flies at a rate of 400 $\frac{km}{h}$ in still air. It can cover 900 km flying with the wind in the same time it takes to fly 700 km against the wind. What is the speed of the wind?

9. Two people are running laps around a 400 m track. They begin at the same point and run in the same direction. The faster person runs at a pace of 4 *minutes* per kilometer, while the slower person runs at 6 *minutes* per kilometer. How long will it take until the faster person laps the slower person?

10. James took a trip, first travelling on a train at 80 $\frac{km}{h}$, then in a car at 90 $\frac{km}{h}$. The entire trip of 265 km took 3 *hours* to complete. If he didnt spend any time waiting between the train ride and the car ride, how long did he spend in each of the two vehicles?