University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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Rates II

Last week, we looked at problems of finding combined rates when two or three people work on an action together. We can generalize this:

If there are n people and person i (with $1 \leq i \leq n$) takes a_i hours to complete the action by themselves, then in one hour they can complete $\frac{1}{a_i}$ of the action. Let T be the amount of time it takes all n people together to complete the action.

We have:

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{1\,hour} = \frac{1}{T}$$

Using this formula we can solve for the amount of time it takes any individual to complete the action or the amount of time it takes all of the people together.

Acceleration:

Acceleration is the rate at which a moving object slows down or speeds up. The quantity it measures against time is a change in speed.

$$Acceleration = \frac{Change\ in\ Speed}{Time}$$

A change in speed has the same units as speed, usually $\frac{m}{s}$ or $\frac{km}{h}$. As a result, the units of acceleration may be $\frac{m}{s^2}$ or $\frac{km}{(h)(s)}$. This looks odd, but you can think of it as $\frac{m}{s}$ per second or $\frac{km}{h}$ per second.

A change in speed is equal to the final speed minus the initial speed of the object. If an object is speeding up, its final speed will be greater than its initial speed, so its change in speed and its acceleration will be positive. If an object is slowing down, its final speed will be less than its initial speed, so its change in speed and its acceleration will be negative.

Example

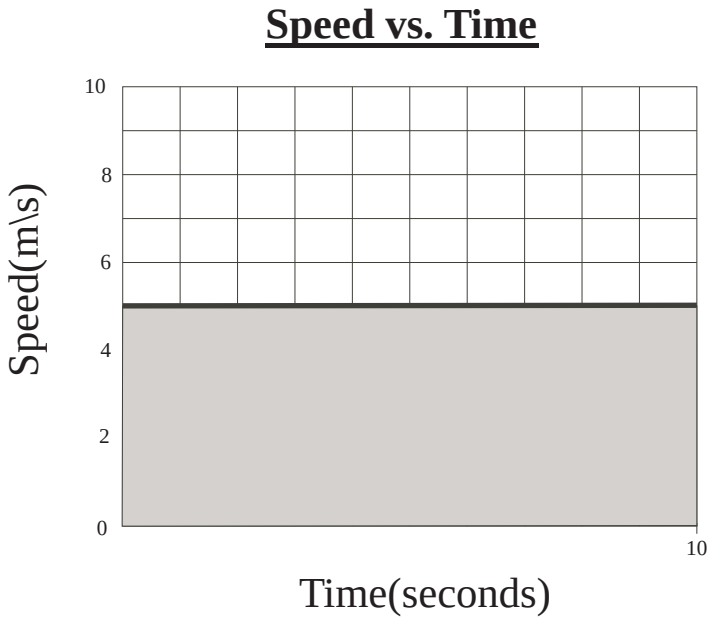
If a car starts at $10\frac{km}{h}$ at accelerates at $5\frac{km}{(h)(s)}$ for 10 *seconds*. What is its final speed? Let us represent the change in speed by $\triangle v$.

$$5\frac{km}{(h)(s)} = \frac{\triangle v}{10s}$$

$$\triangle v = 5\frac{km}{(h)(s)} \times 10s = 50\frac{km}{h}$$

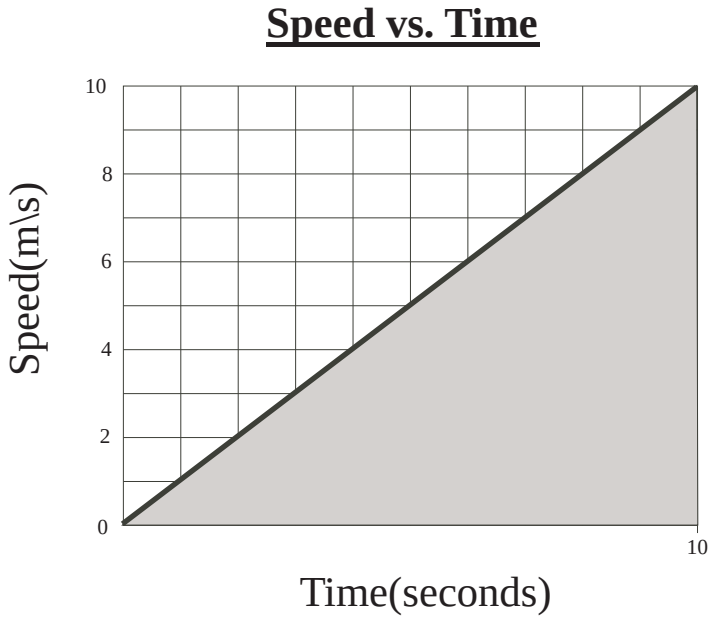
Since the change in speed is $50\frac{km}{h}$ and the car's initial speed is $10\frac{km}{h}$,
its final speed is $10\frac{km}{h} + 50\frac{km}{h} = 60\frac{km}{h}$

Using acceleration and speed or time, you can find distance. Let us look at the case where there is 0 acceleration. The following is a speed-time graph of the situation:



You can see that the speed is $5\frac{m}{s}$ and the time is $10s$. Therefore, the distance is $5\frac{m}{s} \times 10s = 50m$. This is the area of the rectangle since the rectangles length is 10 s and its width is $5\frac{m}{s}$. So, the distance is equal to the area under the speed time graph.

Lets now consider a situation where there is acceleration. The object starts at $0\frac{m}{s}$ and accelerates at $1\frac{m}{s^2}$ for 10 *seconds*. What distance does it travel? We can once again find out by calculating the area under the speed-time graph.



As you can see, in this situation the speed increases at $1\frac{m}{s^2}$ for 10 *seconds* until it reaches $10\frac{m}{s}$ at 10 *seconds*. The slope of this line illustrates the acceleration. The distance is equal to the area under the graph, so

$$d = \frac{1}{2}(10s)\left(10\tfrac{m}{s}\right) = 50m$$

Regardless of the shape of the speed-time graph, the area underneath it is always equal to the distance travelled and the slope of the line at any time is the acceleration at that time.

Problem Set

1. Three bakers: George, Dameon and Lisa have 1.5 hours to make a wedding cake. It usually takes George, Dameon and Lisa 3 hours, 5 hours and 7 hours respectively to complete a wedding cake alone. If they work together, will they have it done in time?

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3. Carly is knitting a scarf. She can complete a scarf in 6 hours, but is interrupted when she is two thirds of the way done. If Melissa, who takes 8 hours to complete a whole scarf, finishes knitting the scarf what was the total amount of time it took to complete the scarf?

4. If a car accelerates from $0\frac{km}{h}$ to $60\frac{km}{h}$ in 5 seconds, what distance does it travel in this time?

5. A car travelling at $150\frac{km}{h}$ passes a police cruiser that is stopped on the side of the road. The police cruiser accelerates at a rate of $10\frac{km}{h}$ per second to a top speed of $160\frac{km}{h}$ and chases the car. At what time after the car passed the police cruiser does the police cruiser catch up to the car?

6. A horse standing at the start line begins to run around a $500m$ track, accelerating at a rate of $0.5\frac{m}{s}$. How long does it take the horse to complete the lap?

7. If a car can accelerate at $3\frac{m}{s^2}$ and can brake at an acceleration of $-5\frac{m}{s^2}$, what is the maximum speed it can reach over a distance of $500m$ if it begins and ends at a stop?

8. Another car can accelerate at $4\frac{m}{s^2}$ and brake at an acceleration of $-5\frac{m}{s^2}$. If the car begins and ends at a stop and travels for 60 seconds, what is the maximum speed it can reach, and what distance did it cover?