University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
October 28, 2009
Counting I

Counting in mathematics is determining the number of ways in which something can occur. For example, counting how many license plates are possible using letters and number in a certain way, finding the number of possible ways of getting a certain hand in a game of cards, or counting how many possible combinations there are on a lock.

The study of the method for counting these kinds of things is called **combinatorics**.

The first thing we will learn, does not only apply to combinatorics, but to all problem solving. The fundamental trick of problem solving is to take a hard problem and turn it into several easier problems.

In combinatorics, two important rules which will help us to do this are the **product rule** and **sum rule**.

Product Rule

If objects consist of two parts where there are m ways of choosing the first part, and n ways of choosing the second part, then there are $m \times n$ objects in total.

Example 1:
Cal Q. Lator has 5 different shirts and 3 different pairs of pants. How many different outfits can he wear?

Solution:
Since there are 5 ways of Cal selecting a shirt and 3 ways of Cal selecting a pair of pants, the product rule tells us that there are $5 \times 3 = 15$ different possible outfits.
Of course, we could have also solved this problem by making a table to list all of the possibilities. Let us do that to check our answer.

Let s_1, s_2, s_3, s_4 and s_5 represent the shirts and let p_1, p_2, and p_3 represent the pants.

$s_1\ p_1$	$s_2\ p_1$	$s_3\ p_1$	$s_4\ p_1$	$s_5\ p_1$
$s_1\ p_2$	$s_2\ p_2$	$s_3\ p_2$	$s_4\ p_2$	$s_5\ p_2$
$s_1\ p_3$	$s_2\ p_3$	$s_3\ p_3$	$s_4\ p_3$	$s_5\ p_3$

<u>Example 2:</u>
A company make both 1" and 2" binders which come in four colours red, green, black or blue. How many different binders do they make?

<u>Solution:</u>
The have 2 sizes and 4 different colours, so the product rule tells us that there are $2 \times 4 = 8$ different possible makes of binder.

<u>Sum Rule:</u>
Suppose the objects in a group are of two types with r objects of the first type and s objects of the second type. If no objects are of both types, then there are $r + s$ objects in the group.

<u>Example 3:</u>
Let's solve example 2 using the sum rule. In the first group the company makes 1" binders which are red, green, black or blue. In the second group the company makes 2" binders which are red, green, black, or blue. Hence, the first type (1" binders) has 4 objects as does the second type (2" binders). Thus the sum rule gives us that there are $4 + 4 = 8$ total objects.

<u>Example 4:</u>
How many two digit positive numbers end in a 5 or 7?

<u>Solution:</u>
What are the two types?

Type 1: End in a 5 - how many of these are there? The numbers have the form $\square 5$, there are 9 choices for the first digit (1,...,9), hence there are 9 objects of this type.

Type 2: End in a 7 - how many of these are there? The numbers have the form $\square 7$, there are 9 choices for the first digit (1,...,9), hence there are 9 objects of this type.

Thus, by the sum rule there are 18 two digit numbers which end in a 5 or 7.

<u>Note:</u> The product rule and the sum rule may be extended to the case where there are more than 2 types/groups.

<u>Example 5:</u>
A restaurant has 4 appetizers, 3 main courses, and 5 desserts. How may different meals could you order?

<u>Solution:</u>
There are 4 ways of selecting an appetizer, 3 ways of select a main course and 5 ways of selecting a dessert, so the product rule gives that there are $4 \times 3 \times 5 = 60$ different possible meals.

<u>Example 6:</u>
How many positive numbers less than 1000 have 1 as the first digit?

<u>Solution:</u>
We have 3 types. 3 digit numbers which have first digit 1, 2 digit numbers which have first digit 1 and 1 digit numbers.

Type 1: These numbers have the form 1xy: There are 10 choices for x and 10 choices for y, hence there are $10 \times 10 = 100$ possible type 1 numbers by the product rule.

Type 2: These numbers have the form 1x: There are 10 choices for x, hence there are 10 possible type 2 numbers.

Type 3: The only type 3 number is 1.

Hence, the sum rule gives that there are 100+10+1=111 numbers less than 1000 which have 1 as the first digit.

<u>Example 7:</u>
How many 2 digit positive numbers are divisible by either 2 or 5?

<u>Solution:</u>
We have two types, divisible by 2 and divisible by 5... but we can not apply the sum rule... why not? Because the sum rule says that we must have no objects which are of both types, but here we have that some numbers are divisible by both 2 and 5. Thus, we have to make sure that we do not count these numbers twice.

Type 1: 2 digit numbers divisible by 2. These numbers have the form xy where x has 9 possibilities and y has 5 possibilities (0,2,4,6,8). Thus, by the product rule, this type has $9 \times 5 = 45$ numbers.

Type 2: 2 digit numbers divisible by 5, but not divisible by 2 (since we counted these in type 1). These numbers have the form x5 and so there are 9 numbers of this type.

Hence, we have that there are $45 + 9 = 54$ numbers that are divisible by 2 or 5.

<u>Note:</u> Another way of doing this problem would be add the number of numbers divisible by 2 and the number of numbers divisible by 5 and then subtract the number of numbers divisible by both 2 and 5.

<u>Excercises</u>

1. Al G. Braw has a 5 math books, 3 science books, and 2 history books.

   a.) If Al wants to read one book, how many choices does he have?

   b.) If Al wants to read one math book, one science book and one history book, how many choices does he have?

2. A vehicle licence plate number consists of 3 letters followed by 3 digits. How many different licence plate are possible?

3. A car can be ordered with 5 choice of colour, 3 choices of upholstery, with or with out air condition, with or without cruise control, and with or without sunroof. How many different cars can be ordered?

4. How many 3 digit positive numbers that are either divisible by 2 or have 3 as the first digit.

Answers

1a	10
1b	$5 \times 3 \times 2 = 30$
2	$26^3 \times 10^3$
3	$5 \times 3 \times 2 \times 2 \times 2 = 120$
4	500
5	810

Permutations

In many of the examples above we were always picking different choices from different sets (colour, size, etc). However, in many cases, we want to pick the items from one set in a particular order.

Example 1:
All the permutations on the number 1,2,3 are:

Notes:

- Order does matter! The permutation 123, is not the same as 321 or 132. In cases, where we don't care about the order we would instead call them **combinations**.
- When we say permutations, we mean that each item from the set may only be used once. However, in some cases we may want to use the items more than once and so we call these **permutations with repetition**.

Can you think of some examples where we would want to use permutations? combinations? permutations with repetition?

Permutations: In how many ways can three potted plants be arranged on a window sill?

Combinations: Picking lottery numbers.

Permutations with repetition: selecting a combination on a combination lock (should be called a permutation lock, since order matters!)

For now, we will just look at permutations and we will look at combinations in the future.

Example 2:
How many permutations are there on n objects?

Solution:
Out of the n-objects, I need to pick one object to be first. I have n choices to do this. Then, since we are not using repetitions, we only have $n-1$ objects left to choose from so we have $n-1$ choices for the second object. Continuing this we see that by the product rule we get that there are $n \times (n-1) \times (n-2) \times \cdots \times (2) \times 1$ permutations on n objects.

Since this product occurs frequently, we give it some notation, called **factorial notation**.

We write $n!$ (which reads " n factorial") and $\mathbf{n! = n \times (n-1) \times (n-2) \times \cdots \times (2) \times 1}$.
For example, $3! = 3 \times 2 \times 1 = 6$, $7! = 7 \times 6 \times 5 \times \cdots \times 2 \times 1 = 5040$.

Example 3:
There are 5 runners in a race. Different prizes are awarded to the runners which finish first, second and third. In how many ways can the prizes be distributed?

Solution:
In this problem we see that we are not looking to count all possible permutations, but rather just the permutations where we select just 3 of the 5 runners.

Let the possible distribution of prizes be represented by abc, where a is the runner who finishes first, b is the runner who finishes second and c is the runner who finishes third. Observe that there are 5 possible runners who could finish first, so there are 5 choices for a. After one runner finishes, there are only 4 remaining who could finish second so we have 4 choices for b. Similarly, we have 3 choices for c. So the number of possible ways of distributing the prizes is $5 \times 4 \times 3 = 60$.

When we are not counting all permutations, but just counting the number of permutations in we which select k objects from n objects, we call these the permutations on n objects taken k at a time.

Excercises

1. Consider the set $\{a, b, c, d\}$.

   a.) How many permutations are there on the set? List them.

   b.) How many permutations on the set have a as the first letter? List them.

   c.) How many permutations on the set have b and c together? List them.

2. How many permutations are there are the numbers 1 - 5 taken 2 at a time. List them.

3. A student club with 10 members wishes to select a president, a secretary and a treasurer from its membership. No member may be selected for more than 1 office. In how many ways can this be done?

4. In how many ways can 25 students be seated in a classroom with 25 desks? With 30 desks?

Answers

1a	$4! = 24$
1b	$3! = 6$
1c	$2 \times 3!$
2	$5 \times 4 = 20$
3	$10 \times 9 \times 8 \times = 720$
4	$25!$, $30 \times 29 \times 28 \times \cdots \times 4$

Problem Set

1. A restaurant menu lists 5 meat dishes and 3 fish dishes.

   a.) How many single course dinners can you order?

   b.) How many dinners can you order than have 1 meat dish and 1 fish dish?

2. How many numbers between 1000 and 9999 have only even digits?

3. A licence plate consists of 4 letters followed by 3 digits. How many different license plates are possible?

4. How many 3 digit numbers are there in which adjacent digits are not the same?

5. In how many was can 6 people seat themselves in a room with 9 chairs where at most 1 person can sit in each chair?

6. How many permutations of the numbers 1, 2, 3, 4, 5, and 6:

   a.) begin with an even number?

   b.) begin with an odd number and end with an even number?

   c.) begin with an odd number and end with an odd number?

7. How many permutations of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 taken 5 at a time:

   a.) have 7 and 8 in adjacent positions

   b.) have 7 and 8 separated by exactly 1 number