University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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Ratios

What is a ratio?

A ratio is a relation of the amounts of two or more things.

A ratio can be expressed in many ways including as a fraction. For example if we have 2 parts coffee for one part cream, we can represent this as a ratio as:

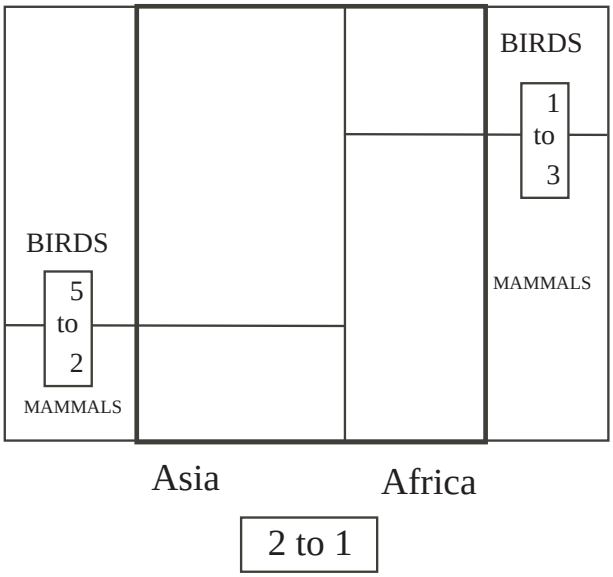
2 to 1 $\qquad$ 2 : 1 $\qquad$ $\frac{2}{1}$

As with fractions, ratios are usually written in lowest form, and one ratio has many equivalent ratios, only one of which is in lowest form.

3 : 1, 1.5 : 0.5, 6 : 2, 12 : 4 and 300 : 100 are all equivalent ratios and 3:1 is the ratio written in its lowest form.

Problem:

A zoo enclosure is divided in the ratio of 2:1 among animals from Asia and animals from Africa. The area of the zoo that contains animals from Asia is divided in the ratio of 5:2 between birds and mammals, while the area that contains animals from Africa is divided in the ratio of 1:3 between birds and mammals. What is the ratio of all birds in the zoo to all mammals in the zoo?



Solution:

Notice that the ratio of animals from Asia to animals from Africa to total animals is $2 : 1 : 3$. Therefore $\frac{2}{3}$ of the animals are from Asia and $\frac{1}{3}$ of the animals are from Africa. Similarly in the Asia enclosure, the ratio of birds to mammals to total animals is $5 : 2 : 7$ so $\frac{5}{7}$ are birds and $\frac{2}{7}$ are mammals. In the Africa enclosure $\frac{1}{4}$ are birds and $\frac{3}{4}$ are mammals.

Therefore, out of the entire enclosure, the fraction of birds is

$$\left(\frac{5}{7} \times \frac{2}{3}\right) + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{10}{21} + \frac{1}{12} = \frac{40}{84} + \frac{7}{84} = \frac{47}{84}$$

while the fraction of mammals is

$$\left(\frac{2}{7} \times \frac{2}{3}\right) + \left(\frac{3}{4} \times \frac{1}{3}\right) = \frac{4}{21} + \frac{1}{4} = \frac{16}{84} + \frac{21}{84} = \frac{37}{84}$$

Therefore the ratio of birds to mammals is $\frac{47}{84} : \frac{37}{84} = 47 : 37$.

The Golden Ratio

The discovery of the Golden Ratio is usually attributed to Pythagoras and his students in Ancient Greece, however the first known written definition of it is by Euclid. Euclid called it the extreme and mean ratio and said a line was cut in this ratio when the ratio of the whole to the larger part was the same as the ratio of the larger part to the smaller.

Using this definition, we can calculate the numerical value of the Golden Ratio. The line below is split into a larger part of length a and a smaller part of length b.

Let us represent the Golden Ratio by phi ϕ.

Then we have $\phi = \frac{a+b}{a} = \frac{a}{b}$. Therefore $b\phi = a$, and we can substitute this into the equation $\phi = \frac{a+b}{a} = \frac{b\phi+b}{b\phi}$. We can then cancel out the bs to get $\phi = \frac{\phi+1}{\phi}$. Multiplying both sides by ϕ and moving everything to one side, we get $\phi^2 - \phi - 1 = 0$

By the quadratic equation, we get

$$\phi = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

Since $a > b$, the golden ratio must be larger than 1, so $\phi = \frac{1+\sqrt{5}}{2}$.

If you take your calculator and calculate this, you will find that the value of the golden ratio is approximately 1.618

The other root of the equation is negative, but when we take its absolute value, we get the golden ratio conjugate $\Phi = \frac{1}{\phi} = \phi - 1$.

The golden ratio can also be written in repeated fraction form as

$$\phi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

The golden ratio has also been linked to the Fibonacci numbers.

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21,... If you take two consecutive Fibonacci numbers, such as 8 and 5 and divide the larger by the smaller, you will get a number close to the golden ratio. As the Fibonacci numbers you choose get higher and higher, the number you get when you divide them will get closer and closer to the golden ratio.

If we let F_n represent the nth Fibonacci number, then this ratio of two consecutive Fibonacci numbers is represented by $\frac{F_n}{F_{n-1}}$. It has been proven that the limit of this ratio as n goes to infinity is exactly the golden ratio. Symbolically,

$$\phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

Problem Set

1. The length and width of the Canadian flag are in the ratio 2 : 1. A company buys a flag with area 10 square metres. What are the dimensions of the flag?

2. A right triangle with sides in the ratio 3 : 4 : 5 has one side of length 60. What are the possible lengths of the other sides?

3. Emilys chemistry teacher instructed her class to make a saltwater solution with a 2 : 3 ratio of salt to water. Emily misheard her teacher and made a 100mL solution with a 2 : 3 ratio of water to salt, instead. How much water must she add to have the correct solution?

4. Evan is 1.8 metres tall. He walks between two lampposts that are 5 metres apart and notices that his shadow from one of the lampposts just touches the base of the other. He also notices that the ratio of his height to the height of the lamppost is 2 to 5. How far is he from each lamppost?

5. A wildlife observer captures 40 deer from a forest, tags them and lets them go back in the forest. Later, he captures 60 deer from the same forest and sees that 18 of them are tagged. What is the approximate number of deer living in the forest? We assume that the ratio of tagged deer to untagged deer in the captured group is the same as the ratio in the entire forest.

6. Kool-Aid is made with flavouring mix, sugar and water. The directions instruct you to mix the flavouring, sugar and water in the ratio 1 : 8 : 32. Jackie decides to make Kool-Aid. She adds 30mL of flavouring and 240mL of sugar to a 1000mL bottle, then fills the rest with water. She wants to drink some of this extra sugary Kool-Aid, and then re-fill the bottle with water to bring it to the proper ratio. How much will she need to drink?

7. In a room, there are a number of people and a number of dogs. The ratio of heads to legs in the room is 5 : 13. Assuming that there are no amputees present, what is the ratio of humans to dogs in the room?

8. Two schools are getting together for a dance. The first school has a 3 : 5 ratio of boys to girls. The second school has a 4 : 3 ratio of boys to girls. All of the students attend the dance and during one song each boy is paired up with one girl. What is the ratio of the number of students at the first school to the number of students at the second school?

9. Kaelee wants to make 75% cocoa chocolate. To do this she mixes together melted 60% cocoa chocolate and 85% cocoa chocolate. If she has 200mL of the 60% cocoa chocolate, how much 85% chocolate will she need?

10. Quick Shipping Co. will only ship boxes that have the sum of their length, width and height under 240cm. If a box has its length with and height in the ratio 3 : 2 : 5, and it is small enough to be shipped, what is its maximum possible volume?

11. (AMC 12A 2003) Al, Bert and Carl are the winners of a school drawing for a pile of Halloween candy, which they are to divide in a ratio of 3 : 2 : 1, respectively. Due to some confusion, they come at different times to claim their prizes, and each assumes he is the first to arrive. If each takes what he believes to be the correct share of candy, what fraction of the candy goes unclaimed?