University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
February 10, 2010
Number Theory II

Opening Problem

At CEMC High School, all of the students are 13-19 years old. Each teacher has to submit a class number, which is found by multiplying together the ages of every student in the class.

1. Mr. Crone's class number was 757856736000. How many students are in his class and how old is each student?

2. Can the office always tell from the class number how many students are in a class and what their ages are? Explain why or why not.

Greatest Common Divisor

Definition: The *greatest common divisor* or *gcd* of two or more numbers is the largest number that is a factor of each of the numbers. This can also be called the greatest common factor.

Example Find the greatest common divisor of 30 and 24.

30 has factors: 1 2 3 5 6 10 15 30
24 has factors: 1 2 3 4 6 8 12 24
The factors that both numbers have in common are 1, 2, 3 and 6.
Of these, 6 is the largest so the greatest common divisor of 30 and 24 is 6.

Exercise 1 List the factors then find the greatest common divisor of:
1) 15 and 49

15: 1 3 5 15
49: 1 7 49

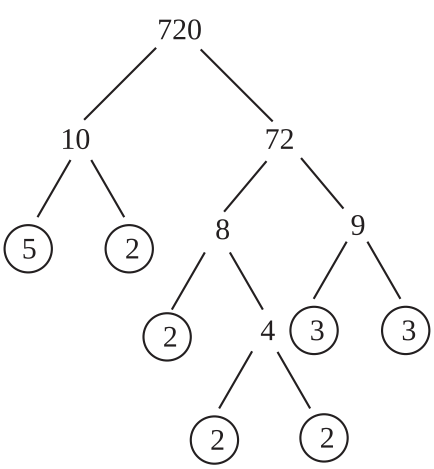
2) 6 and 42

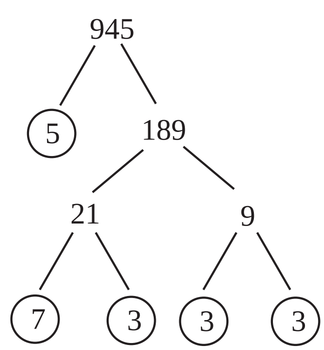
6: 1 2 3 6
42:1 2 3 6 7 14 21 42

Example Find the greatest common divisor of 720 and 945.

These numbers are very large, and it would take too long to list all of their factors.

Instead, we are going to look at the factor tree of each. This is another method to finding the greatest common divisor.



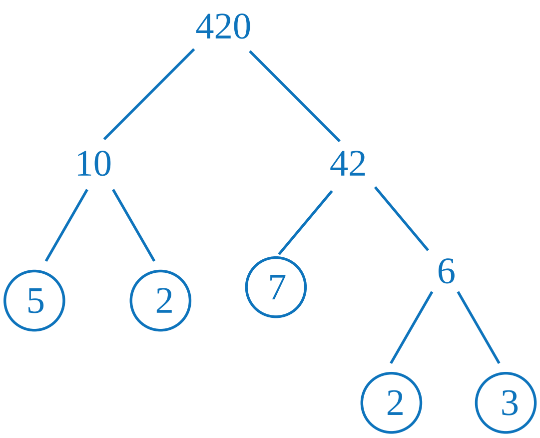


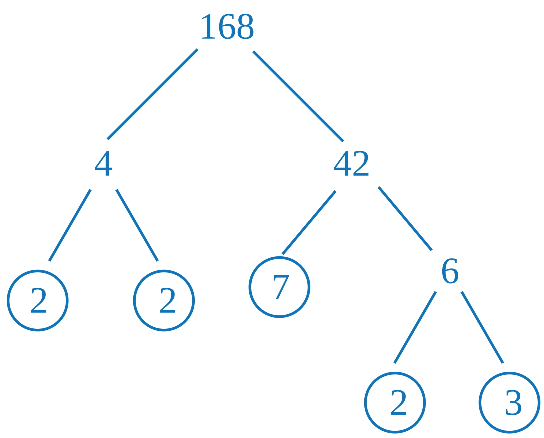
$720 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \qquad 945 = 3 \times 3 \times 3 \times 5 \times 7$

To find the greatest common divisor of these numbers, we look at all of the prime factors they have in common. Both numbers have two 3s and one 5 as prime factors. Therefore, their greatest common divisor is $3 \times 3 \times 5 = 45$.

Exercise 2 Find the greatest common divisor of the following numbers using factor trees.

1) 420 and 168



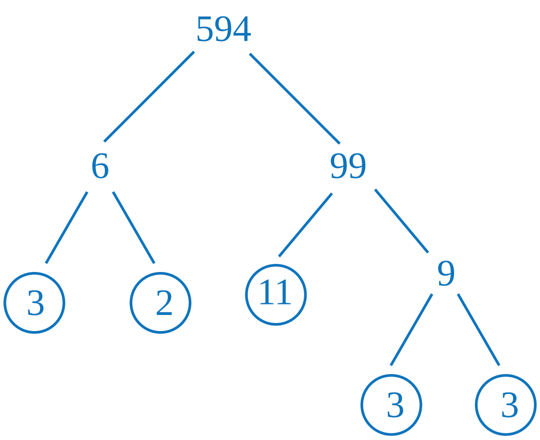


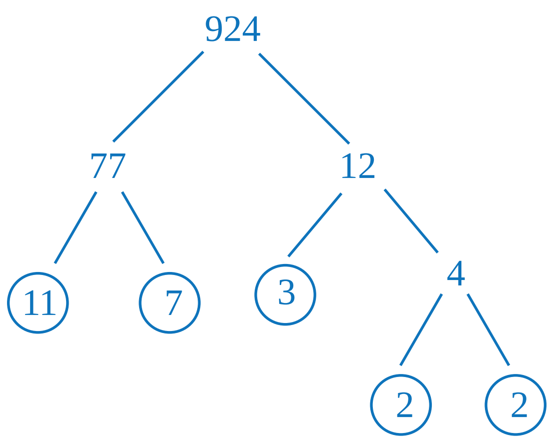
$420 = \underline{2} \times 2 \times \underline{3} \times 5 \times \underline{7}$

$168 = \underline{2} \times 2 \times 2 \times \underline{3} \times \underline{7}$

gcd $(420{,}168) = 2 \times 3 \times 7 = 42$

2) 594 and 924





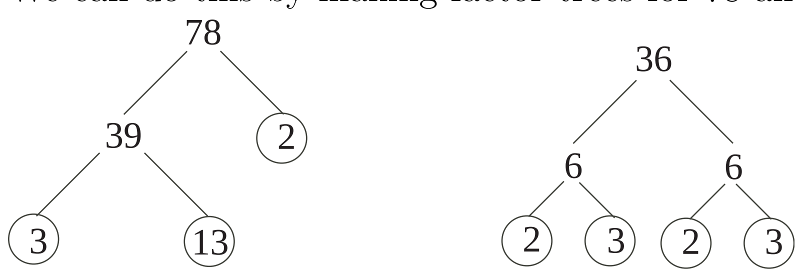
$594 = \underline{2} \times \underline{3} \times 3 \times 3 \times \underline{11}$

$924 = \underline{2} \times 2 \times \underline{3} \times 7 \times \underline{11}$

gcd $(594{,}924) = 2 \times 3 \times 11$

Example Mike has 78 baseball cards and 36 baseballs that he plans to sell in packages. Each package must contain the same number of cards and the same number of baseballs. What is the greatest number of packages he can sell so that there are no leftover cards or baseballs?

Since we need to divide the 78 baseball cards into the packages with none left-over, the number of packages must be a divisor of 78. Similarly, the baseballs must be divided up evenly so the number of packages must be a divisor of 36. Since we are looking for the *greatest* number of packages, we want to find the greatest common divisor of 78 and 36.
We can do this by making factor trees for 78 and 36.



The greatest common factor is 6. Therefore, Mike can sell 6 packages with $78 \div 6 = 13$ baseball cards and $36 \div 6 = 6$ baseballs in each.

Least Common Multiple

Definition: A *multiple* of a number is the product of the number and another whole number. Factors and multiples are closely related. For example, 9 is a factor of 90, so 90 is a multiple of 9.

Definition: The *least common multiple* or *lcm* of two or more numbers is the lowest number that is a multiple of each of the numbers.

One method of finding the least common multiple involves listing the multiples of each number until you find the first number that is a multiple of both.

Example Find the least common multiple of 6 and 8.

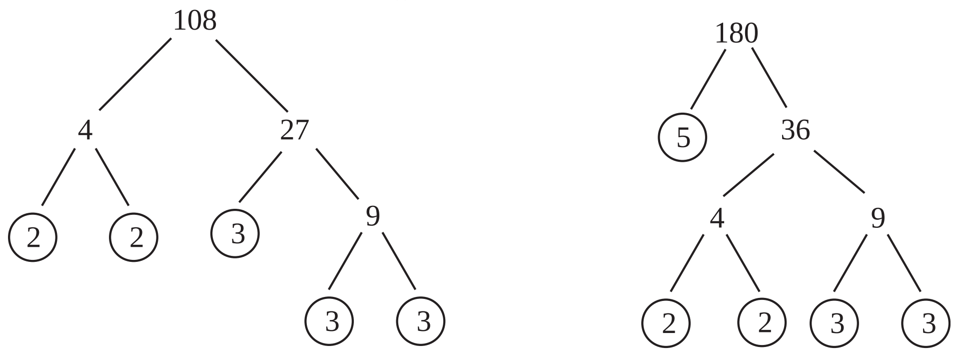
The first few multiples of 6 are 6 12 18 $\underline{24}$ 30 36 42 48...
The first few multiples of 8 are 8 16 $\underline{24}$ 32 40 48 56 64...

The first multiple that occurs in both is 24, so the least common multiple is 24.

This method may take too long if we are dealing with larger numbers. Instead, we will use another method involving prime factors.

Example Find the least common multiple of 108 and 180.

Since these numbers are large, we will use the method of prime factors to find the least common multiple.

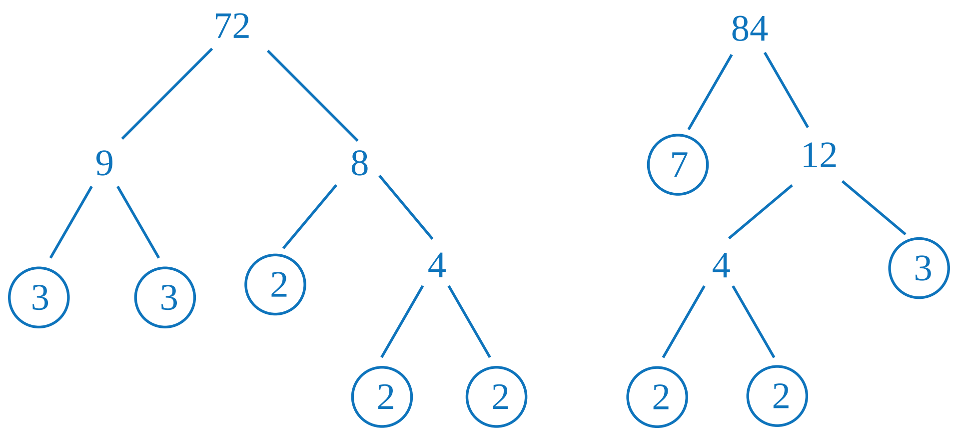


$108 = 2 \times 2 \times 3 \times 3 \times 3$ $\quad$ $180 = 2 \times 2 \times 3 \times 3 \times 5$

The least common multiple must include all of the prime factors that occur in each of the numbers. Therefore we need two 2s, three 3s and one 5.
Therefore, the least common multiple is $2 \times 2 \times 3 \times 3 \times 3 \times 5 = 540$.

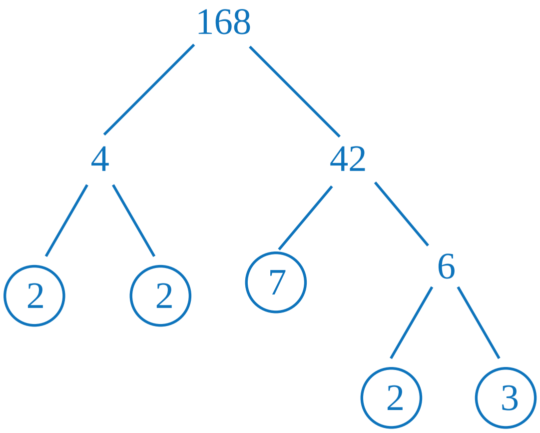
Notice that by looking at the prime factors we can tell that $540 = 108 \times 5$ and $540 = 180 \times 2 \times 3$.

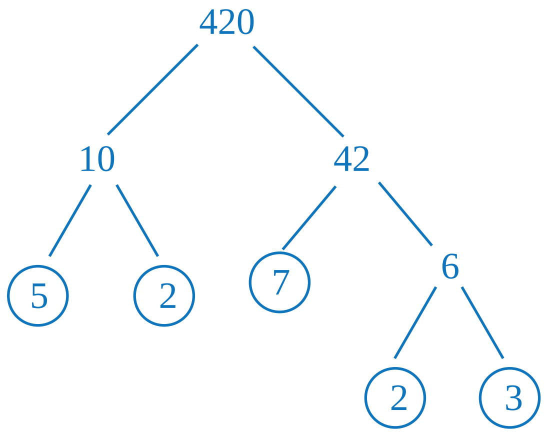
Exercise 3 Find the least common multiple of the following numbers using factor trees.
1) 72 and 84



$72 = 2 \times 2 \times 2 \times 3 \times 3$
$84 = 2 \times 2 \times 3 \times 7$
$\text{lcm}\ (72, 84) = 2 \times 2 \times 2 \times 3 \times 3 \times 7 = 504$

2) 168 and 420



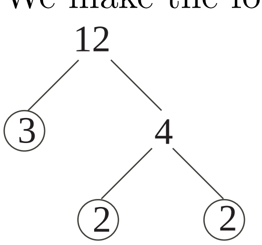


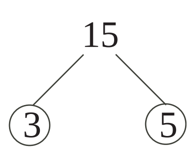
$168 = 2 \times 2 \times 2 \times 3 \times 7$
$420 = 2 \times 2 \times 3 \times 5 \times 7$
$\text{lcm}\ (168{,}420) = 2 \times 2 \times 2 \times 3 \times 5 \times 7 = 840$

Example A snack bar at the Sky Tower ordered bags of chips, which they received in packages of 12 each. A snack bar at the Air Cloud Center ordered the same number of bags of chips, but received them in packages of 15 each. What is the lowest number of bags the snack bars could have ordered, and how many packages did each receive?

The number of bags ordered must be a multiple of 12, since the Sky Tower received them in packages of 12. Similarly, it must also be a multiple of 15. Since we are looking for the least number of bags, we want to find the least common multiple of 12 and 15.

We make the following factor trees:





So the least common multiple of 12 and 15 is $2 \times 2 \times 3 \times 5 = 60$. Therefore the Sky Tower and the Air Cloud Center each ordered 60 bags of chips. The Sky Tower received $60 \div 12 = 5$ packages, and the Air Cloud Center received $60 \div 15 = 4$ packages.

Exercise 4 Complete the following chart.

A	B	C	D	E	F
First Number	Second Number	gcd of A and B	lcm of A and B	$A \times B$	$E \div C$
495	945	45	10395	467775	10395
168	234	6	6552	39312	6552
520	189	1	98280	98280	98280
345	765	15	17595	263925	17595

1. What do you notice about the relationship between column D and column F?
2. Write this relationship in the form of an equation.

1. The number in columns D and F are the same.
2. The equation is: $\text{lcm (A,B)} = \dfrac{\text{A} \times \text{B}}{\text{gcd (A,B)}}$

Problem Set

1. Sam was buying hot dogs and hot dog buns for a backyard barbeque. Hot Dogs come in packs of 16, but buns come in packs of 12. How many packs of each will Sam have to buy so that there are no hot dogs or buns left over?

2. A florist has 72 roses, 84 tulips and 48 orchids that she wants to use to create bouquets. What is the largest number of identical bouquets she can put together without having any flowers left over?

3. Three alarm clocks are set off at the same time. If the first one beeps every 9 seconds, the second one beeps every 12 seconds and the third one beeps every 15 seconds, how long will it be until they beep at the same time?

4. Vince has three pieces of rope with lengths of 300 cm, 312 cm and 396 cm. He wants to cut the three pieces of rope into smaller pieces of equal length with none left over.

   (a) What is the greatest possible length of each of the smaller pieces of rope?

   (b) How many of the smaller pieces of rope will he have altogether?

5. Two flashing signs are turned on at the same time. One sign flashes every 4 seconds and the other flashes every 6 seconds. How many times will they flash at the same time in 1 minute?

6. Valerie is cutting a huge block of cheese into smaller cubes for her kids' snack time. If the block of cheese measures 15 cm by 30 cm by 42 cm and each cube of cheese has a whole number of centimetres as a side length, what is the size of the largest cubes she can cut the block into so there is no leftover cheese?

7. Joe lists the following fractions: $\frac{1}{56}, \frac{2}{56}, \ldots, \frac{55}{56}, \frac{56}{56}$. He then crosses out any of them that are not in lowest form. How many fractions are left uncrossed once he is done?

8. (a) When is the least common multiple of two numbers a and b equal to $a \times b$?

   (b) When is the least common multiple of two numbers a and b equal to either a or b?

9. The gcd of two numbers is 30, and the lcm is 840. If one of the numbers is 120, what is the other number?

10. If a number a is a factor of b and a factor of c, is a a factor of $b + c$? Explain.

Answers

1. 3 packs of hot dogs and 4 packs of buns
2. 12 bouquets each consisting of 6 roses, 7 tulips, and 4 orchids
3. 180 seconds
4. (a) 12 cm
   (b) 84 smaller pieces of rope
5. 5 times
6. 3 cm × 3 cm × 3 cm
7. 24
8. (a) when $\gcd(a, b) = 1$
   (b) when a divides b or b divides a
9. 210
10. Yes.

Opening Problem

1. There are 10 kids in Mr. Crone's class, with ages 13, 14, 14, 15, 15, 15, 16, 17, 18 and 18.
2. $13 = 13$ $\quad 14 = 2\times7$ $\quad 15 = 3\times5$ $\quad 16 = 2\times2\times2\times2$
   $17 = 17$ $\quad 18 = 2 \times 3 \times 3$ $\quad 19 = 19$

   Yes, the office will always be able to tell. Once the class number is prime factored, the number of times 13, 7, 5, 17 and 19 appear will reveal the number of 13, 14, 15, 17 and 19 year olds, respectively. The number of 14 and 15 year olds will also remove some factors of 2 and 3 from the prime factorization. Next, from the number of 3's left in the prime factorization, the number of 18 year olds can be found. Finally there will only be 2's left, which will reveal the number of 16 year olds.

   Notice that the office can only do this because the age range has been restricted to 13-19 years old. If we included 12 year olds or 20 year olds, it would not always be possible.