University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
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Exponents

Opening Problem

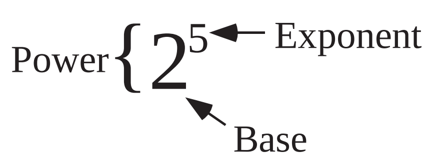
My rich uncle gave me a dollar for my 3rd birthday. On each birthday after that, he tripled his previous gift. How much money did I receive from my uncle on my 8th birthday?

Solution Each year the gift my uncle gives me triples, which means the new gift is found by multiplying the previous year's gift by 3. So, on my 4th birthday my uncle gave me $1 \times 3 = 3$ dollars, on my 5th birthday he gave me $1 \times 3 \times 3 = 3 \times 3$ dollars, and so on.
Therefore, on my 8th birthday he gave me $3 \times 3 \times 3 \times 3 \times 3 = 243$ dollars.
An easier way to write this is using a power: $3^5 = 243$.

Powers

Definition: A *power* is a single numerical expression that means repeated multiplication. For example, the expression $2 \times 2 \times 2 \times 2 \times 2$ can be written as the power 2^5.

A power has a *base* and an *exponent*. The *base* is the number that is multiplied repeatedtly and the *exponent* is the number of times the base is multiplied as shown below:



The number 2^5 is read "2 to the exponent 5" or "2 to the 5th". For simplicity, a number raised to the exponent 2 is often referred to as "squared", so 10^2 is read "10 squared". Also, a number raised to the exponent 3 is often referred to as "cubed", so 7^3 is read "7 cubed".

Exercise 1 Write out each expression in exponential form, then evaluate.

1) $4 \times 4 \times 4 \times 4 \times 4$	2) $13 \times 13 \times 13 \times 13$	3) $8 \times 8 \times 3 \times 3 \times 3 \times 3$
$4^5 = 1024$	$13^4 = 28561$	$8^2 \times 3^4 = 5184$
4) 6	5) $9 \times 9 \times 9 \times 2 \times 2$	6) $1 \times 1 \times 1$
$6^1 = 6$	$9^3 \times 2^2 = 2916$	$1^3 = 1$

Multiplying and Dividing Powers

An expression such as $x^a \times x^b$ is a multiplication of powers. These expressions can be simplified <u>if both powers have the same base</u>.

Example Simplify the expression $6^4 \times 6^2$.

First, expand both powers:	$(6 \times 6 \times 6 \times 6) \times (6 \times 6)$
Then, drop the brackets:	$6 \times 6 \times 6 \times 6 \times 6 \times 6$
Now, simplify as one power:	6^6

An expression such as $x^a \div x^b$ is a division of powers. These expressions can also be simplified <u>if both powers have the same base</u>.

Example Simplify the expression $10^9 \div 10^5$.

First, expand both powers:	$\dfrac{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10 \times 10}$
Then, divide out repeated 10s:	$\dfrac{\cancel{10} \times \cancel{10} \times \cancel{10} \times \cancel{10} \times \cancel{10} \times 10 \times 10 \times 10 \times 10}{\cancel{10} \times \cancel{10} \times \cancel{10} \times \cancel{10} \times \cancel{10}}$
Simplify the expression:	$10 \times 10 \times 10 \times 10$
Now, simplify as one power:	10^4

Exercise 2 Complete the table.

Product/Quotient of Powers	Product/Quotient Form	Power Form
$5^5 \times 5^3$	$(5 \times 5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5)$	5^8
$9^7 \div 9^6$	$\frac{9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9}{9 \times 9 \times 9 \times 9 \times 9 \times 9}$	$9^1 = 9$
$11^4 \times 11^3$	$(11 \times 11 \times 11 \times 11) \times (11 \times 11 \times 11)$	11^7
$4^2 \times 6^3 \times 4^2 \times 6^1$	$(4 \times 4 \times 6 \times 6 \times 6) \times (4 \times 4 \times 6)$	$4^4 \times 6^4$
$(2^3 \times 7^5) \div (2^2 \times 7^1)$	$\frac{2 \times 2 \times 2 \times 7 \times 7 \times 7 \times 7 \times 7}{2 \times 2 \times 7}$	2×7^4

Did you notice a pattern while completing the table?

When multiplying powers, we can obtain a single power by adding the exponents.
When dividing powers, we can obtain a single power by subtracting the exponents.

Power of Powers

An expression such as $(x^a)^b$ is a power of a power. These expressions can be simplified using what we learned above.

Example Simplify the expression $(7^4)^3$ as a single power.

First, expand the outer exponent:	$7^4 \times 7^4 \times 7^4$
Use the rule for multiplying powers:	$7^{(4+4+4)}$
Now, simplify as one power:	7^{12}

Exercise Simplify the following expressions as a single power.

1) $(3^6)^4$ 2) $(15^5)^1$ 3) $(9^3 \times 4^2)^2$

3^{24} 15^5 $9^6 \times 4^4$

Did you notice a pattern while completing the exercise?

When we have a power of a power, we can obtain a simplified single power by multiplying the two exponents.

Exponent Laws		
Multiplying Powers:	$x^a \times x^b = x^{a+b}$	
Dividing Powers:	$x^a \div x^b = x^{a-b}$	where $a > b$
Power of a Power:	$(x^a)^b = x^{(a \times b)}$	

Zero & Negative Exponents

Powers may not always have a positive exponent. It is possible to have an exponent values that are zero or negative.An expression such as x^a where $a < 0$ is a power with a negative exponent. An expression such as x^a where $a = 0$ is a power with a zero exponent.

For example, simplifying the expression $6^5 \div 6^5$ to a single power would give us 6^0 using the exponent laws above. Similarly, simplifying the expression $4^8 \div 4^{10}$ would give us 4^{-2}.

Example Complete the following table from 2^1 upwards first. Then, using the pattern, complete the rest of the table. Use fractions instead of decimals.

Power	Value
2^3	8
2^2	4
2^1	2
2^0	1
2^{-1}	$\frac{1}{2}$
2^{-2}	$\frac{1}{4}$
2^{-3}	$\frac{1}{8}$

We can see that a pattern arises as we start from the top of the table and work our way downwards: each step we divide the previous value by 2.

From this, we can see that 2^0 has a value of 1. This is the same for all powers: any base (except 0) raised to the exponent 0 has a value of 1.

Also, we can see that when we continue the pattern into negative exponents, we can rewrite all the denominators of the fractions as a power of 2.

$$2^{-1} = \frac{1}{2} = \frac{1}{2^1} \qquad 2^{-2} = \frac{1}{4} = \frac{1}{2^2} \qquad 2^{-3} = \frac{1}{8} = \frac{1}{2^3}$$

This is the same for all powers: any base raised to a negative exponent can be rewritten as a fraction where the denominator is the base raised to the positive value of that exponent.

Exercise 4 Write the following powers as a power with positive exponents.

1) 3^{-8} 2) $(9^{-3})^4$ 3) $(5^{-6} \times 5^2)^0$

$\frac{1}{3^8}$ $\frac{1}{9^{12}}$ $5^0 = 1$

More Exponent Laws		
Zero Exponent:	$x^0 = 1$	
Negative Exponents:	$x^{-n} = \frac{1}{x^n}$	where $n > 0$

Problem Set

1. A population of squirrels triples each year. If there are 50 squirrels now, how many will there be 5 years from now?

2. Alannah puts \$0.01 into a bank account which will double her money each day. She needs \$20.00 to buy a toy. How many days will it take until she has enough money?

3. How many of the integers between 2 and 50 can be written as a power x^y, where x and y are positive integers and $y > 1$? (eg. $8 = 2^3 : x = 2, y = 3$)

4. Britney Gallivan was the first person to fold a piece of paper in half 12 times, something which had previously been believed to be impossible. How many layers of paper would be in that stack?

5. A nanometre is equal to 10^{-9} metres and a millimetre is equal to 10^{-3} metres. How many nanometres are equal to one millimetre?

6. (a) If $2^a \times 2^b = 128$, what is $a + b$?

   (b) If $3^c \div 3^d = 9$, what is $c - d$?

7. Fill the four boxes below with the numbers $1, 2, 3$ and 4 so that the expression has the largest possible value.

$$\square^{\square} \times \square^{\square}$$

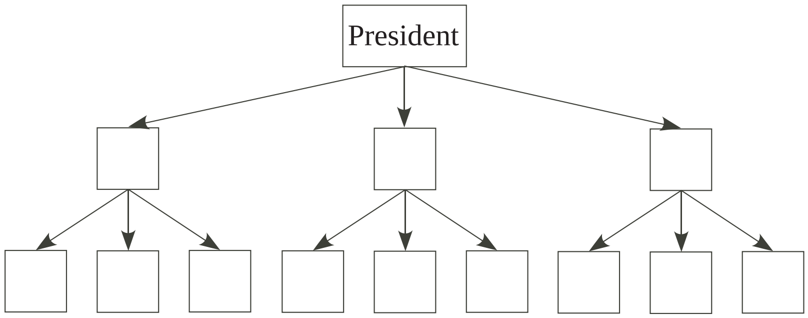
8. Without using a calculator, write the following numbers in order from least to greatest.

   3^{55} $\quad 4^{44}$ $\quad 5^{33}$

9. Monster Corporation has a phone tree in case of an emergency. It starts with the president. In the first round of calls, he calls three employees. In the second round of calls, each of those three employees calls three more employees, and so on.

   (a) How many new people are called in the fifth round of calls?

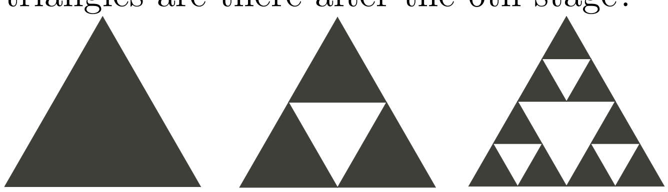
   (b) If the Corporation has 3000 employees, including the President, how many rounds of calls will it take to notify all of them of the emergency?



10. Ramone dumped a large bucket of water onto his driveway to rinse it off. The water from the bucket covered twice as much area every 2 seconds after it initially hit the ground. After 20 seconds, the whole driveway was covered. How long did it take the water to cover half of the driveway?

11. A chemical called carbon-14 has a half life of about 5730 years, which means that every 5730 years, the amount of carbon-14 in a specimen halves. An animal skull was found that is known to have only $\frac{1}{32}$ of the amount of carbon-14 as when the animal died. How long ago did the animal die?

12. Below we see the first couple stages of Sierpinski's Triangle. In each stage a triangle $\frac{1}{4}$ of the size of each existing black triangle is cut out from the centre of the black triangle. If the original large black triangle has an area of 100cm^2, what is the area of the each triangle removed in the 6th stage? How many black triangles are there after the 6th stage?



Stage 1 Stage 2

13. Find the number that goes in the box. (*Hint*: $4 = 2 \times 2 = 2^2$)

$$4^2 \times 8^3 = 2^{\square}$$

14. Find the value of $a^{3b} - 5$ if $a^b = 2$.

15. Find $\dfrac{15^3}{5^2}$ without calculating 15^3.

16. Find the value of $\dfrac{2^{2010} - 2^{2009}}{2^{2010} + 2^{2009}}$. (Hint: Use your algebra skills to factor the expression.)

17. If $(2^3 \times 2^x) + 2^x = (3^2 \times 3^y) - 3^y$ where x and y are positive integers, what are x and y? (Hint: Factor)

Answers

1. 12150 squirrels
2. 11 days
3. 8. They are $4(2^2), 8(2^3), 9(3^2), 16(4^2 \text{ or } 2^4), 25(5^2), 27(3^3), 32(2^5), 36(6^2)$ and $49(7^2)$.
4. 4096 layers
5. 10^6 nanometres are equal to 1 millimetre.
6. (a) $a + b = 7$
   (b) $c - d = 2$
7. $3^4 \times 2^1 = 162$
8. Least to Greatest: $5^{33}, 3^{55}, 4^{44}$
9. (a) 243 people
   (b) 7 rounds of calls
10. 18 seconds
11. 28650 years ago
12. At the 6th stage there are 729 black triangles and the area of each triangle removed is 0.024 cm^2.
13. 13
14. 3
15. $3^3 \times 5 = 135$
16. 3^{-1} or $\dfrac{1}{3}$
17. $x = 3, y = 2$