University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
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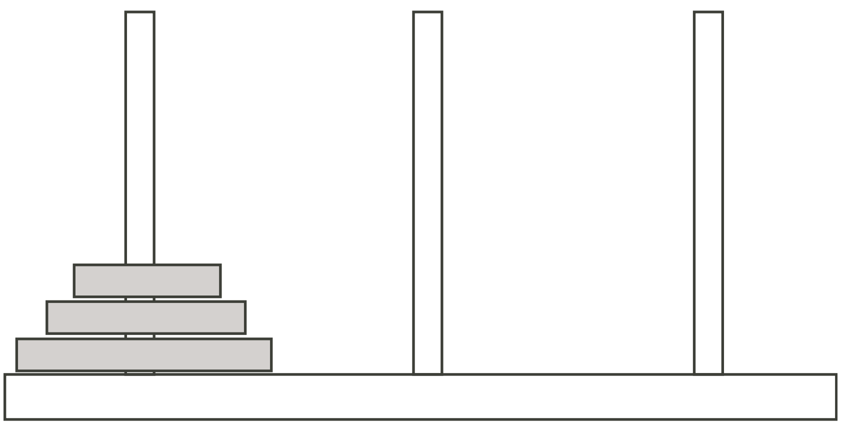
Patterns

Opening Problem: Tower of Hanoi



You have three wooden pegs and three rings stacked on the first peg, each ring slightly smaller than the one below it. You want to move the stack of rings to the third peg. However, you may only move one ring at a time, and you may never place a larger ring on top of a smaller one.
What is the minimum number of moves required? 7

You have the same three pegs, but now there are four rings stacked on the first peg. What is the minimum number of moves to move the stack to the third peg?

Number Patterns

Defintion: A *pattern* is a collection of objects that are related by a rule. The rule can be a written desciption or a formula.

Let's look at the following number pattern:

$1, 3, 5, 7, 9, 11, 13, \ldots$

We could describe it by saying that these are the odd positive integers from least to greatest. We could also say that the pattern starts with 1, then the next number is found by adding 2.

In this lesson, we are going to use a_n to mean the nth number in a pattern.

So for the pattern description above, we could say that $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, and so on. Then, what is a_n? Since we said above that each term is equal to the term before it plus two, we can write a_n as $a_n = a_{n-1} + 2$.

This is a *recursive* definition of a pattern, since the definition of each term is based on previous terms.

We could also find a rule that does not need any previous terms in the definition. For this pattern we can see that the nth term is one less than the nth multiple of 2. Therefore, we can define the pattern by the rule $a_n = 2n - 1$.

This is an *explicit* definition of a pattern, since the definition of each term is based only on the term number, and does not depend on other terms.

Exercise 1

Describe the following patterns recursively and explicitly, and use your rules to determine the 7th and 8th terms.

1. $1, 4, 7, 10, 13, 16, \ldots$
   Recursive: $a_1 = 1, a_n = a_{n-1} + 3$
   $a_7 = a_6 + 3 = 16 + 3 = 19$
   $a_8 = a_7 + 3 = 19 + 3 = 22$
   Explicit: $a_n = 3n - 2$

$a_7 = 3(7) - 2 = 21 - 2 = 19$
$a_8 = 3(8) - 2 = 24 - 2 = 22$

2. $100, 95, 90, 85, 80, 75, \ldots$
   Recursive: $a_1 = 100, a_n = a_{n-1} - 5$
   $a_7 = a_6 - 5 = 75 - 5 = 70$
   $a_8 = a_7 - 5 = 70 - 5 = 65$
   Explicit: $a_n = 105 - 5n$
   $a_7 = 105 - 5(7) = 105 - 35 = 70$
   $a_8 = 105 - 5(8) = 105 - 40 = 65$

3. $1, 2, 4, 7, 11, 16, \ldots$
   Recursive: $a_1 = 1, a_n = a_{n-1} + n - 1$
   $a_7 = a_6 + 7 - 1 = 16 + 7 - 1 = 22$
   $a_8 = a_7 + 8 - 1 = 22 + 8 - 1 = 29$
   Explicit: $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$
   $a_7 = \frac{1}{2}(7^2) - \frac{1}{2}(7) + 1 = \frac{49}{2} - \frac{7}{2} + 1 = 22$
   $a_8 = \frac{1}{2}(8^2) - \frac{1}{2}(8) + 1 = 32 - 4 + 1 = 29$

Sometimes it is easier to describe a pattern recursively, but not so easy to describe it explicitly.

$1, 1, 2, 3, 5, 8, \ldots$

Can you determine the rule for this pattern? What is it?
Add the previous two terms to get the next term.

What are the next three numbers in the pattern?
13, 21 and 34

This pattern is called the *Fibonacci sequence* and can be defined recursively:

$$a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}$$

However, it is very difficult to find an explicit formula for the Fibonacci sequence. The formula below is an explicit formula:

$$a_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \text{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

As we can see, this is an extremely complicated formula. As long as we do not want a term very far along in the sequence, it is probably easier to use our recursive definition above.

The opposite can also be true: a pattern can be described easily by an explicit definition but not so easily by a recursive one. Take a look at the following pattern:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

What is an explicit definition of this pattern?
$a_n = \frac{1}{n}$

Can you find a recursive definition of this pattern?
$a_1 = 1, a_n = \dfrac{n-1}{n} \times a_{n-1}$ OR $a_1 = 1, a_n = a_{n-1} - \dfrac{1}{n(n-1)}$

As you can see, the recursive definiton is not so simple as the explicit one you found above. For this pattern, finding the recursive definiton involves a lot of algebra. The formula is:

$$a_1 = 1, a_n = \frac{n-1}{n} \times a_{n-1}$$

The advantage of an explicit definition is that we can find a number in the pattern without needing to find all the numbers that come before it.

Example Find the 60th term in the pattern $1, 3, 5, 7, 9, 11, 13, \ldots$.

We already found an explicit formula for this pattern: $a_n = 2n - 1$.
We want to find the 60th term, which is a_{60} so we substitute $n = 60$ into the formula.
We get $a_{60} = 2(60) - 1 = 120 - 1 = 119$.
Therefore, the 60th term in the pattern is 119.

Exercise 2 Using your explicit definitions of the patterns in Exercise 1, find the 20th and 50th term in each pattern.

1. $a_n = 3n - 2$
   $a_{20} = 3(20) - 2 = 60 - 2 = 58$
   $a_{50} = 3(50) - 2 = 150 - 2 = 148$

2. $a_n = 105 - 5n$
   $a_{20} = 105 - 5(20) = 105 - 100 = 5$
   $a_{50} = 105 - 5(50) = 105 - 250 = -145$

3. $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$
   $a_{20} = \frac{1}{2}(20^2) - \frac{1}{2}(20) + 1 = 200 - 10 + 1 = 191$
   $a_{50} = \frac{1}{2}(50^2) - \frac{1}{2}(50) + 1 = 1250 - 25 + 1 = 1226$

Patterns in Exponents

Example Find the units digit of 4^{55}.

4^{55} is such a large number that we cannot simply use a calculator to look at the units digit. Instead, let's see if we can find a pattern.
$4^1 = 4$ which has units digit 4
$4^2 = 16$ which has units digit 6
$4^3 = 64$ which has units digit 4
$4^4 = 256$ which has units digit 6
We can see that the unit's digit of powers of 4 follows the pattern $4, 6, 4, 6, \ldots$. If the exponent is odd, the units digit will be 4 and if the exponent is even, the units digit will be 6.
Since 55 is odd, 4^{55} will have units digit 4.

Exercise 3 What is the units digit of 3^{4287}?
$3^1 = 3$ $\quad$ $3^2 = 9$ $\quad$ $3^3 = 27$ $\quad$ $3^4 = 81$ $\quad$ $3^5 = 243$
The pattern seems to be the digits $3, 9, 7, 1$ repeated.
$4287 = 4(1071) + 3$ so 4287 falls in the 3rd place in the pattern.
Therefore, the units digit of 3^{4287} is 7.

Fractals

Definition: A *fractal* is a geometric pattern involving self-similarity. *Self-similarity* is when smaller parts of a shape are similar to the whole shape.

Fractals occur in many places in nature. One example is a fern:

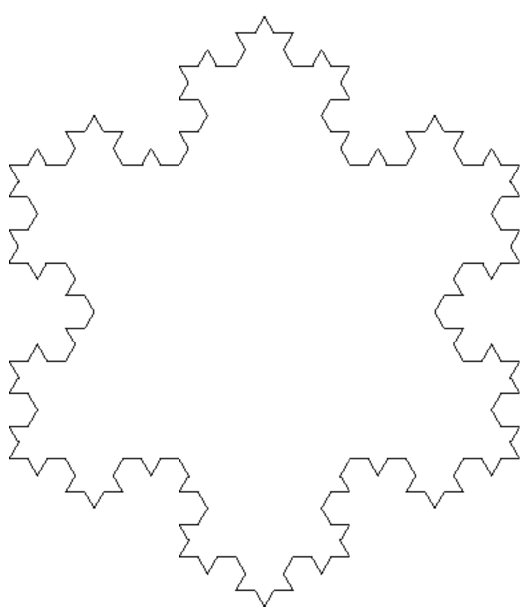
A fern has groups of leaves branching off on either side of the main stem. These leaves once again have groups of leaves branching off of a main stem. Even the smallest size leaves we can see on the fern have a bumpy shape that is similar to the entire fern.

What are other examples of fractals in nature?
blood vessels, broccoli, mountain ranges, nautilus shells

Exercise 4 Follow the steps below to create a fractal called the Koch Snowflake.

1. Draw a fairly large equilateral triangle on a sheet of paper.
2. Divide each line segment of the triangle into three equal sections.
3. Draw two lines the same length as the middle section, to form an equilateral triangle pointing outwards, with the middle section as the base.
4. Erase the middle section, leaving the triangle you just drew with only two sides.
5. Repeat the above steps with each of the smaller line segments in your new picture.

Your drawing should look something like the one below:



What is the perimeter of the Koch Snowflake?

To figure this out, let's look at the perimeter at each stage.
In the first stage, the perimeter is just the perimeter of the equilateral triangle.

In the second stage, we take out the middle $\frac{1}{3}$ of each side, so we are removing $\frac{1}{3}$ of the perimeter.
However, we then draw two sides of a triangle in each of these holes. Each of the sides of the triangle is the length of $\frac{1}{3}$ of the side of the original triangle, so we are adding back $2 \times \frac{1}{3} = \frac{2}{3}$ of the original perimeter.

Assume our original perimeter was P. Our new perimeter is:

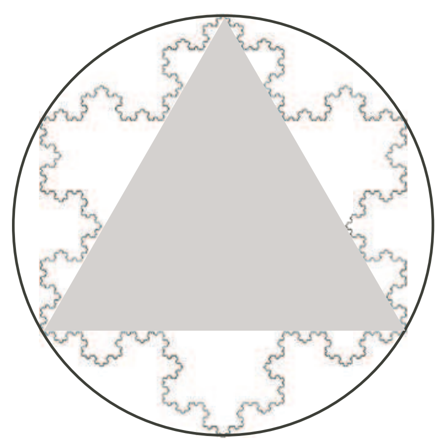
$$P - \frac{1}{3}P + \frac{2}{3}P = \left(\frac{3}{3} - \frac{1}{3} + \frac{2}{3}\right) P = \frac{4}{3}P$$

In the next stage, we do the same thing. We remove $\frac{1}{3}$ of each side, then add back $\frac{2}{3}$ of the length of each side. So now the perimeter is $\frac{4}{3}\left(\frac{4}{3}P\right) = \frac{16}{9}P$.

This process continues, and at each stage the perimeter is multiplied by $\frac{4}{3}$. The Koch snowflake is the figure that is made after an infinite number of stages, and since the perimeter is increasing each time, the perimeter of the Koch snowflake is infinite.

What is the area of the Koch Snowflake?

If we drew a circle around the original equilateral triangle which passed through each of the vertices of the triangle, we would see that the Koch snowflake never goes outside this area. Therefore, we know that the area of the Koch snowflake is not infinite, since it must be less than or equal to the area of the circle.



In fact, the area of the Koch snowflake is equal to $\frac{8}{5}A$ where A is the area of the original equilateral triangle.

Problem Set

1. Determine the rule of the following sequence:

   $$1, 5, 9, 13, 17, 21, 25, \ldots$$

   Using the rule,

   (a) find the 28th number in the sequence.

   (b) find the 316th number in the sequence.

2. Which of the numbers from 1 - 9 has the same units digit when that number is taken to any exponent $x, x > 0$?

3. Determine the rule for the following pattern and fill in the blanks below:

   2, __ , 8, 16, 32, __ , __ , 256, 512, ...

4. What is the units digit of 2^{63}? 2^{1568}?

5. What would be the first negative number in the pattern below and when does it occur?

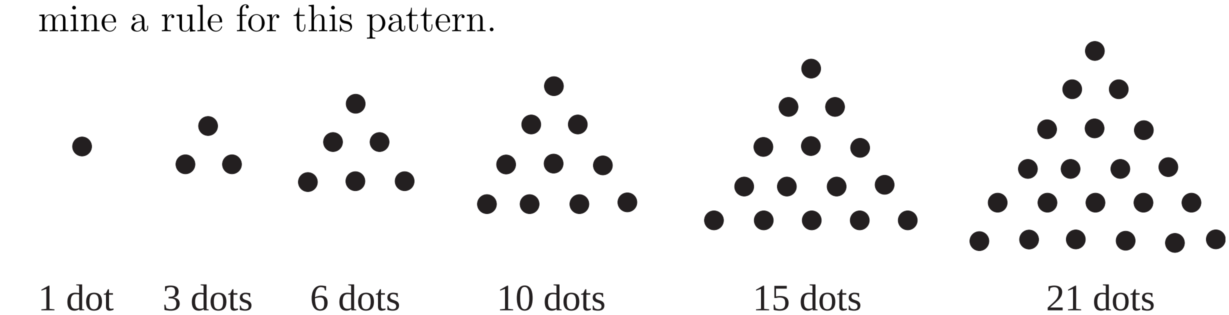
   $$200, 193, 186, 179, 172, \ldots$$

6. (a) Using your calculator, find the decimal value of $\frac{5}{27}$.

   (b) What is the 200th digit after the decimal place in the decimal expansion of $\frac{5}{27}$?

   (c) What is the 333rd digit after the decimal place in the decimal expansion of $\frac{1}{7}$?

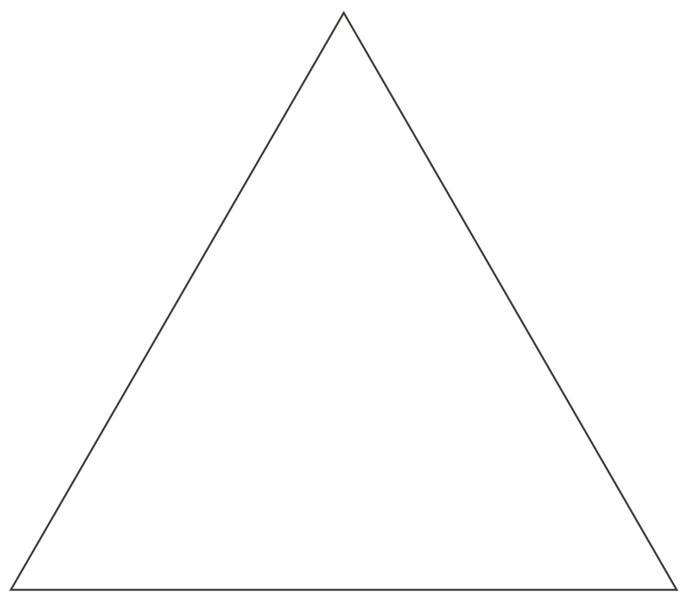
7. The pattern below is generated by making a triangular pattern of dots. Determine a rule for this pattern.



8. We draw an equilateral triangle with side length 9 cm, and use this to create a Koch Snowflake. What is the perimeter of the figure after the 5th stage? (Remember, the equilateral triangle is the 1st stage!)

9. (a) What is the sum of the first 3 odd numbers? the first 4? the first 5?
   (b) Determine the rule for this pattern.
   (c) What is the sum of the first 50 odd numbers?

10. Follow the steps below to create a fractal called Sierpinski's Triangle.
    (a) Find the midpoints of each of the sides of the white triangle(s) below.
    (b) Connect the midpoints to make an upside-down equilateral triangle, and colour this triangle in.
    (c) You will now see three white triangles around the middle coloured triangle. Repeat steps (a) and (b) in each of these triangles.
    (d) Continue this process until the white triangles left are too small to divide further.



11. **Challenge:** The following pattern is sometimes called the "Look-and-say sequence".

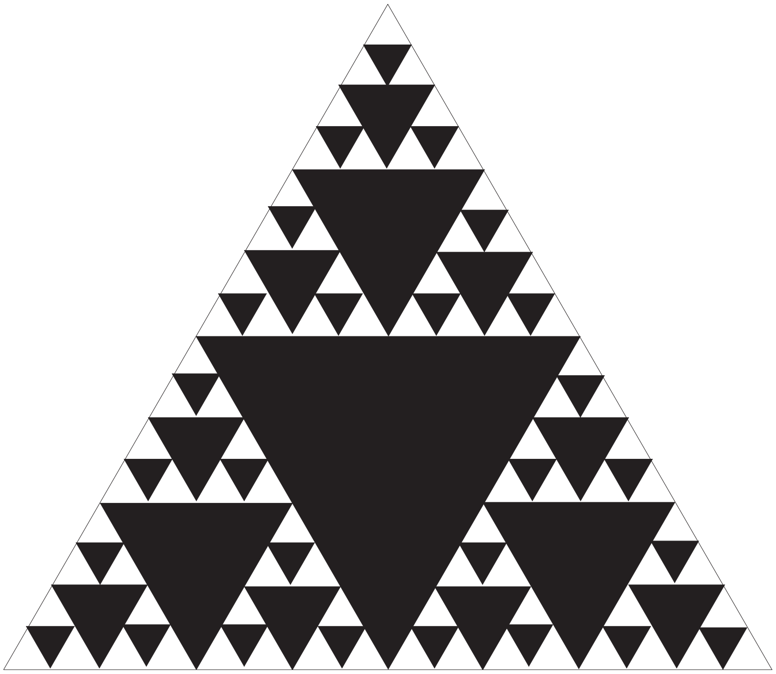
$$1, 11, 21, 1211, 111221, 312211, \ldots$$

    (a) What is the rule for this pattern?
    (b) Will the number 4 ever appear as a digit in this sequence?

Answers:

1. $a_1 = 1, a_n = a_{n-1} + 4$
   $a_n = 4n - 3$
   (a) $a_{28} = 4(28) - 3 = 112 - 3 = 109$
   (b) $a_{316} = 4(316) - 3 = 1264 - 3 = 1261$
2. 1, 5 and 6
3. $a_1 = 1, a_n = a_{n-1} \times 2$
   $a_n = 2^n$
   $2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$
4. 2^{63} ends in an 8.
   2^{1568} ends in a 6.
5. The first negative number is -3 and it is the 30th term.
6. (a) $\frac{5}{27} = 0.185185185\ldots$
   (b) The 200th digit is 8.
   (c) $\frac{1}{7} = 0.142857142857\ldots$ so the 333rd digit is 2.
7. $a_1 = 1, a_n = a_{n-1} + n$
   $a_n = \frac{1}{2}n^2 + \frac{1}{2}n$
8. $P = (27)(\frac{4}{3})^4 \approx 85.3$ cm
9. (a) $1 + 3 + 5 = 9 \quad 1 + 3 + 5 + 7 = 16 \quad 1 + 3 + 5 + 7 + 9 = 25$
   (b) The sum of the first n odd numbers is n^2.
   (c) 2500

10.



11. (a) Each term describes the one before it verbally.
The second term desribes the first term which is “one one” or 11. So the second term is 11.
The third term describes the second term which is “two ones” or 21. So the third term is 21.
The fourth term desrcibes the third one which is “one two one one” or 1211. So the fourth term is 1211.
This pattern continues on this way.

(b) No. Think about why.