University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

# Junior Math Circles
# February 3, 2010
# Number Theory I

## Factors

Definition: A *factor* of a number is a whole number that divides evenly into the other number. So, when you divide by the factor the remainder is zero.

Factors come in pairs. When you divide a number by one of its factors, the answer or quotient is the paired factor.

Is 4 a factor of 36?

Yes, 36 divided by 4 gives us an answer of 9 with no remainder.

Is 3 a factor of 47?

No, 47 divided by 3 gives us an answer of 15 with a remainder of 2.

**Exercise 1** State all the factors of the following numbers.

| 1) 50: | 2) 15: | 3) 13: | 4) 36: |
|---|---|---|---|
| $1, 2, 5, 10, 25, 50$ | $1, 3, 5, 15$ | $1, 13$ | $1, 2, 3, 4, 6, 9, 12, 18, 36$ |

Notice that 36 has an odd number of factors. Each factor still has a pair, but 6 pairs with itself, and is only listed once.

**Divisibility Rules**

The rules listed in the table below help us determine whether a number is divisible by each of the numbers from 2 to 11. Note that 0 is divisible by all positive numbers.

| **Divisible by** | **Test** | **Example** |
|---|---|---|
| 2 | The ones digit is even. | 46 is divisible by 2 since 6 is even. |
| 3 | The sum of all the digits is divisible by 3. | 153 is divisible by 3 since $1+5+3 = 9$, which is divisible by 3. |
| 4 | The last two digits are divisible by 4. | 820 is divisible by 4 since 20 is divisible by 4. |
| 5 | The ones digit is 0 or 5. | 795 is divisible by 5 since its last digit is 5. |
| 6 | The number is divisible by 2 and 3. | 258 is divisible by 6 since 8 is even so it is divisible by 2 and $2+5+8 = 15$ which is divisible by 3. |
| 7 | Subtract twice the ones digit from the rest of the number, until you get a small number that is divisible by 7. | 672 is divisible by 7 since $2 \times 2 = 4$ and $67 - 4 = 63$, which is divisible by 7. |
| 8 | The last three digits are divisible by 8. | 4816 is divisible by 8 since 816 is divisible by 8. |
| 9 | The sum of all the digits is divisible by 9. | 567 is divisible by 9 since $5+6+7 = 18$ which is divisible by 9. |
| 10 | The ones digit is 0. | 270 is divisible by 10 since the ones digit is 0. |
| 11 | The alternating sum of the digits is divisible by 11. | 1364 is divisible by 11 since 1-3+6-4=0, which is divisible by 11. |

**Exercise 2** Using the rules above, check off each of the numbers from 2 to 11 that each number is divisible by.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 926 | ✓ | | | | | | | | | |
| 1420 | ✓ | | ✓ | ✓ | | | | | ✓ | |
| 7848 | ✓ | ✓ | ✓ | | ✓ | | ✓ | ✓ | | |
| 22176 | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | | ✓ |
| 39208 | ✓ | | ✓ | | | | ✓ | | | |

**Prime Numbers**

Definitions:
A *prime number* is a number that has only two factors, 1 and itself.

A *composite number* it is a number with factors other than 1 and itself.

Exceptional Case: The number 1 is neither prime nor composite by definition.

**Exercise 3** Determine whether the following numbers are prime or composite. If it is composite, list out its factors.

| | |
|---|---|
| 1) 19 | 2) 22 |
| prime | composite - $1, 2, 11, 22$ |
| 3) 37 | 4) 51 |
| prime | composite - $1, 3, 17, 51$ |

Are there any even numbers that are prime?

Yes, 2 is the only prime number that is even. Think about why.

How can we find prime numbers?

We can find them using the method called **Sieve of Eratosthenes**.

Instructions:

1. Cross off 1 since it is not prime.
2. Find the first prime number (2) and circle it.
3. Go through the rest of the chart and cross off all of the multiples of two.
4. Go back to the beginning of the chart and find the first number that is not crossed off. Circle it, it is prime. Then go through the rest of the chart, crossing off the multiples of this number.
5. Repeat step 4, until every number in the chart is either circled or crossed off.
6. The numbers that are circled are the prime numbers.

**Exercise 4** Use this method to find all the prime numbers from 1 to 100.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

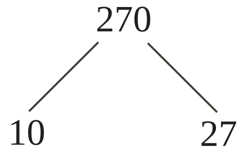
Write down the primes less than 100. There should be 25 of them.
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$

**Prime Factorization**

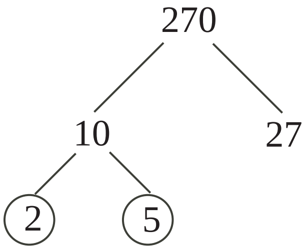
One way of describing numbers is by breaking them down into a product of their prime factors. This is called prime factorization. Every positive number can be prime factored. By definition the prime factorization of a prime number is the number itself, and the prime factorization of 1 is 1.

**Example** Let's prime factor 270 using a factor tree.

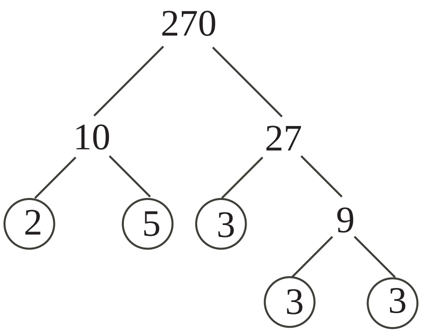
**Solution** First think of a factor of 270. We don't want to use 1 or 270.
Let's use 10.
Now what factor is paired with 10? That factor is 27, since $10 \times 27 = 270$.



Is 10 prime? No, so we have to prime factor 10.
What is a pair of factors of 10? A pair is 2 and 5.
Both 2 and 5 are prime, so we are done this part.




Is 27 prime? No, so we have to prime factor it.
What is a pair of factors? One pair is 3 and 9.
The number 3 is prime, but 9 has factors 3 and 3.

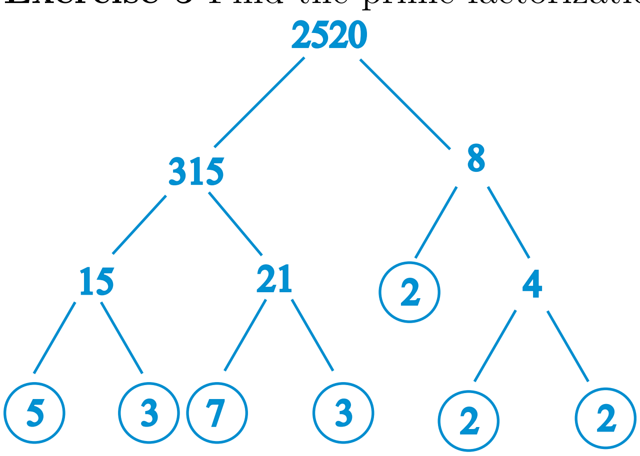



We have decomposed 270 into all prime numbers, so now we can write it as a product of these primes.

$$270 = 2 \times 3 \times 3 \times 3 \times 5$$

Note: Prime factorizations are unique. Every number has only one prime factorization, and no two numbers have the same prime factorization.

**Exercise 5** Find the prime factorization of 2520 using a factor tree.



**Problem Set**

1. Ms. Timson wants to split her Grade 9 math class of 36 students into groups for an upcoming assignment. List all the possibilities of groups, each with the same number of students, that Ms. Timson can divide her class into so that no students are left without a group.

2. What is the first prime number greater than 200?

3. Draw two different factor trees for the number 100. (Start with two different pairs of factors.)

4. The volume of a cereal box is 1925 $\text{cm}^3$. What are the different possible dimensions of the cereal box?
(Note: The volume of a rectangular box is *length* × *width* × *height*.)

5. The eight digit number 1234□678 is divisible by 11. What is the digit □?

6. The four digit number 43□□ is divisible by 3, 4 and 5. What are the last two digits?

7. (a) If a number is divisible by 2 and 3, is it always divisible by 6?
   (b) If a number is divisible by 2 and 4, is it always divisible by 8?

8. What is the smallest number that you must multiply 48 by so that the product is divisible by 45?

9. The product of three *different* positive integers is 144. What is the maximum possible sum of these three integers?

10. The digits $1, 2, 3, 4, 5$ and 6 are each used once to compose a six digit number $abcdef$, such that the three digit number $abc$ is divisible by 4, $bcd$ is divisible by 5, $cde$ is divisible by 3 and $def$ is divisible by 11. Determine all possible assignments of the digits to the letters.

11. What is the smallest number that you must multiply 1512 by in order to get a perfect square.
A *perfect square* is a number that is equal to another number multiplied by itself. For example $6 \times 6 = 36$ is a perfect square.

12. If $x$ and $y$ are two-digit positive integers with $xy = 555$, what is $x + y$?

## Answers

1. The possibilities are: 1 group of 36 students, 2 groups of 18 students, 3 groups of 12 students, 4 groups of 9 students, 6 groups of 6 students, 9 groups of 4 students, 12 groups of 3 students, 18 groups of 2 students and 36 groups of 1 student.

2. 211

3. 


4. The different possible dimensions are:

| | |
|---|---|
| 1cm $\times$ 1 cm $\times$ 1925 cm | 1cm $\times$ 35 cm $\times$ 55cm |
| 1 cm $\times$ 5 cm $\times$ 385 cm | 5 cm $\times$ 5 cm $\times$ 77 cm |
| 1cm $\times$ 7 cm $\times$ 275 cm | 5 cm $\times$ 7 cm $\times$ 55 cm |
| 1 cm $\times$ 11 cm $\times$ 175 cm | 5 cm $\times$ 11 cm $\times$ 35 cm |
| 1 cm $\times$ 25 cm $\times$ 77 cm | 7 cm $\times$ 11 cm $\times$ 25 cm |

5. 9

6. 20 or 80 - The number could be 4320 or 4380.

7. (a) Yes, by definition, if a number is divisible by 2 and by 3 then it is divisible by 6.
   (b) No, for example, in the exercise in the lesson above, 1420 is divisible by 4 and by 2 but it is not divisible by 8.

8. 15.

9. 75. The three divisors are 72, 2, and 1.

10. $abcdef = 324561$

11. 42

12. $x = 15$ and $y = 37$, so $x + y = 52$.