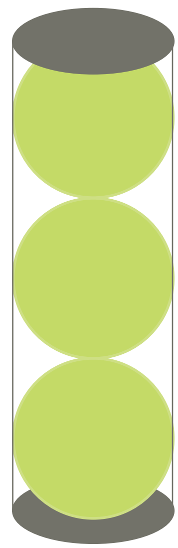
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
March 10, 2010
3D Geometry II

Opening Problem



Three tennis ball are packed in a cylinder. The balls at the top and bottom are touching the top and bottom of the cylinder, respectively, and all three balls are touching the sides of the cylinder. The tennis balls have a radius of 3 cm. What is the volume of air in the tube?

Surface Area of 3D Figures

Definition: The *surface area* SA of a 3D figure is the sum of the areas of all of its faces and curved surfaces. The surface area of any figure is measured in square units (or units squared).

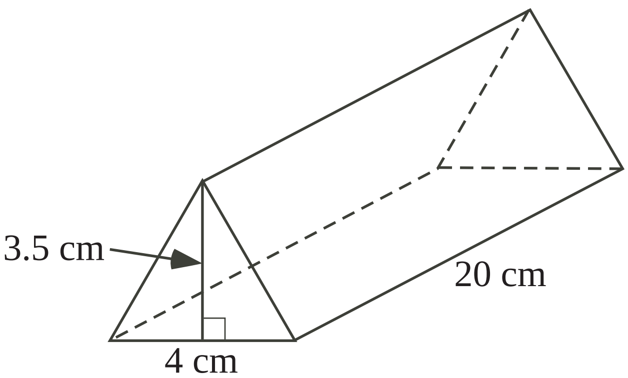
Surface Area of a Prism

The surface area of a prism is equal to the area of both bases plus the area of each face that joins the two bases together. Each of the faces that joins the two bases together will have a width equal to one of the sides of the base and length equal to the height of the prism.

$$SA = (2 \times \text{Area of Base}) + (h \times \text{Perimeter of Base})$$

+ Note that for a cylinder, the perimeter of the base is actually the circumference of the circle, $2\pi r$.

Example A chocolate bar is packaged in the cardboard equilateral triangular prism shown below. What area of cardboard is required to create the box?

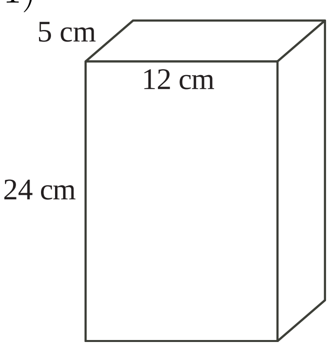


We know that this shape is a prism with an equilateral triangle base.
The area of the triangle is $\frac{1}{2} \times 3.5$ cm$\times 4$ cm$=7$ cm^2.
Since the triangle is equilateral, all of its sides measure 4 cm,
so its perimeter is 3×4 cm$=$ 12 cm.

The surface area of the prism is therefore
$(2 \times 7\text{ cm}^2) + (20\text{ cm} \times 12\text{ cm}) = 14\text{ cm}^2 + 240\text{ cm}^2 = 254\text{ cm}^2$.
So, 254 cm^2 of cardboard is needed to create the box.

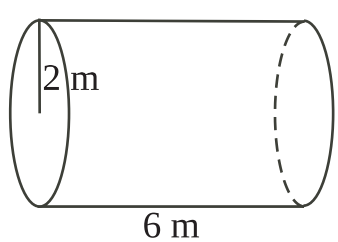
Exercise 1 Find the surface area of the following prisms.

1)



$$\begin{aligned} SA &= 2(5 \times 12 + 24 \times 12 + 5 \times 24) \\ &= 2(60 + 288 + 120) \\ &= 936\text{ cm}^2 \end{aligned}$$

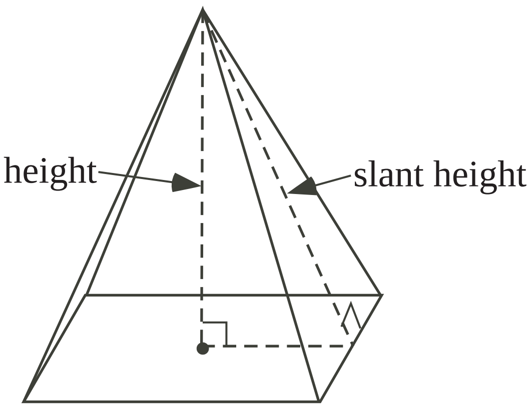
2)



$$\begin{aligned} SA &= 2 \times \pi \times 2^2 + 6 \times 2 \times \pi \times 2 \\ &= 8\pi + 24\pi \\ &= 32\pi \\ &\approx 100.5\text{ m}^2 \end{aligned}$$

Surface Area of a Pyramid

Definition: The *slant height*, s, of a triangular face of a pyramid is the height of the triangle, running from the base to the common vertex. This is different from the height as shown below.



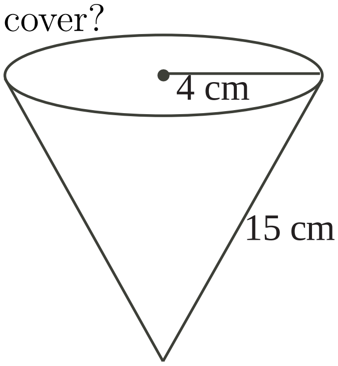
The surface area of a pyramid is equal to the area of the base plus the area of each triangular face that meets at the common vertex. The area of each triangular face is equal to its slant height times the length of the side of the base.

$$SA = \text{Area of Base} + \text{Area of Triangular Faces}$$

For a cone, the slant height is the length of a line from the common vertex to any point on the edge of the circular base. For a cone,

$$SA = \pi rs + \pi r^2$$

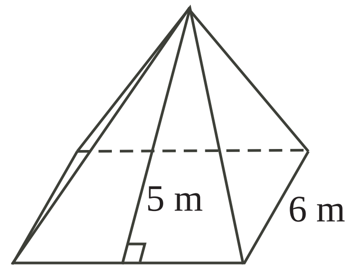
Example An ice cream cone with a radius of 4 cm and a slant height of 15 cm is going to be dipped in chocolate. What is the surface area that the chocolate will cover?



An ice cream cone does not have a flat base like other cones, and so we do not need to include πr^2 in the surface area, since this is the area of the circular base.

Therefore, the surface area of the cone will be πrs.
$SA = \pi \times 4$ cm$\times$15 cm$= \pi \times 60$ cm$^2 \approx 188.5$ cm^2.

Exercise 2
The pyramid below has 4 identical triangular faces and a square base. Find its surface area.



$$\begin{aligned} SA &= 6^2 + 4(\tfrac{1}{2} \times 5 \times 6) \\ &= 36 + 60 \\ &= 96 \text{ m}^2 \end{aligned}$$

Surface Area of a Sphere

To find the surface area of a sphere, try this activity at home.

1. Find an orange that is approximately spherical. Measure its circumference at the widest point and use this to find its radius. On a sheet of paper, draw circles with the same radius.

2. The area of each of these circles is πr^2. How many of these areas do you think is equal to the surface area of the orange? 2? 3? 4? 5? More?

3. Peel the orange, and lay the pieces of peel inside one of the circles you drew. Once you fill a circle, move on to another one. How many circles did the orange peel cover?

The orange peel covers 4 circles. This means that the surface area of a sphere is equal to four times the area of a circle with the same radius.

The surface area of a sphere is:

$$SA = 4\pi r^2$$

Example A ping pong ball has a radius of 2 cm. What was the area of plastic required to create the ball?

A ping pong ball is a sphere, and the plastic is a thin layer forming the suface, so we want to find the surface area of a sphere with radius 2 cm.

$SA = 4\pi \times (2 \text{ cm})^2 = 4\pi \times 4 \text{ cm}^2 \approx 50.3 \text{ cm}^2$

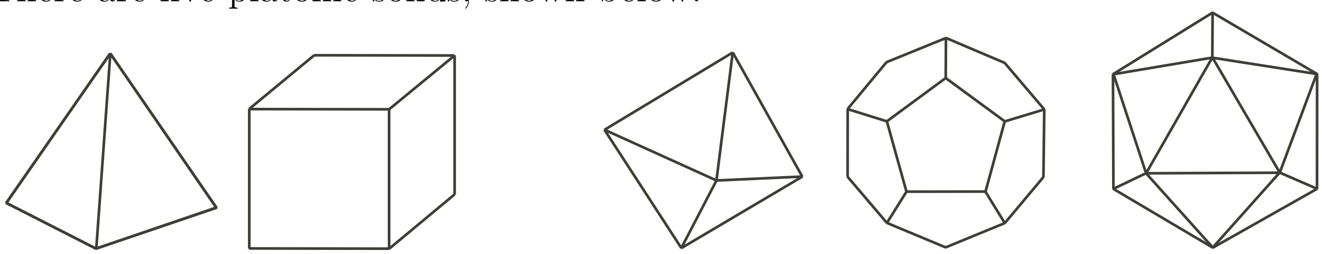
So 50.3 cm^2 of plastic was required to make the ball.

Platonic Solids

Definition: A *regular polygon* is a polygon with all equal lenght sides and all equal interior angles. For example, equilateral triangles and squares are both regular polygons.

Definition: *Platonic solids* are polyhedra that have congruent regular polygons as faces, and the same number of faces meeting at each vertex.

There are five platonic solids, shown below:



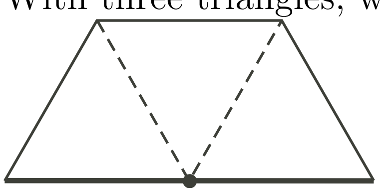
Tetrahedron Hexahedron (Cube) Octahedron Dodecahedron Icosahedron

Why are there only five platonic solids?

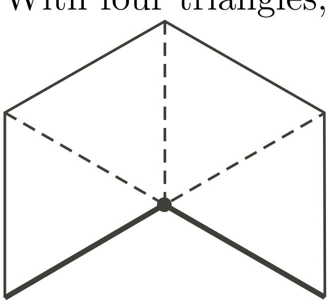
Let's look at a single vertex of a platonic solid and the faces surrounding it, unfolded. The vertex will have at least three faces around it, and these faces will be congruent regular polygons.

In the diagrams below the dotted lines are fold lines and the thick lines will be glued together to form a 3D vertex.

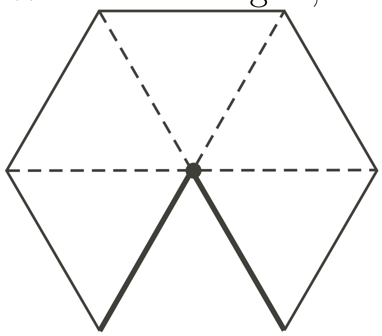
Let's look at triangular faces:
With three triangles, we get a vertex of a tetrahedron.



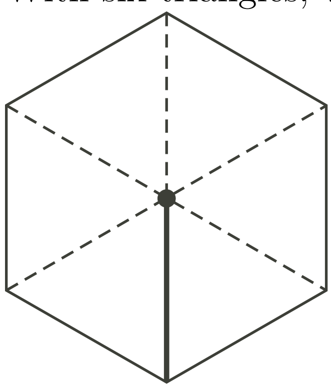
With four triangles, we get a vertex of an octahedron.



With five triangles, we get a vertex of an icosahedron.

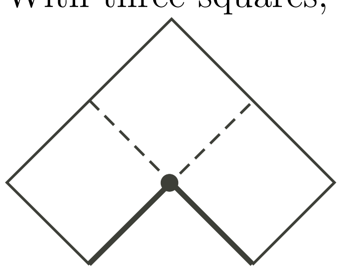


With six triangles, we get the configuration below.



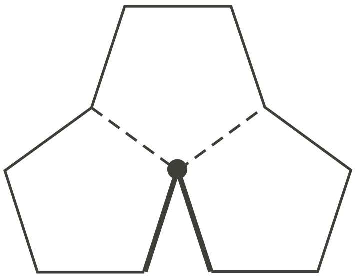
This cannot possibly fold up into a vertex, because it is perfectly flat. The interior angle in each triangle is $60°$ and $6 \times 60° = 360°$, which is a full circle.

Now let's look at square faces:
With three squares, we get a vertex of a cube.



With four squares, the sum of angles in the middle would be $4 \times 90° = 360°$, so we cannot create a 3D shape this way.

Now let's look at pentagonal faces:
With three pentagons, we get a vertex of a dodecahedron.

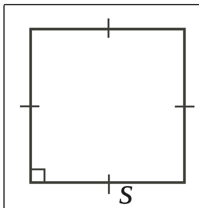
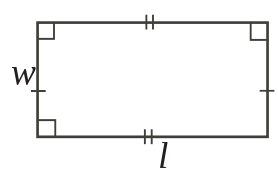
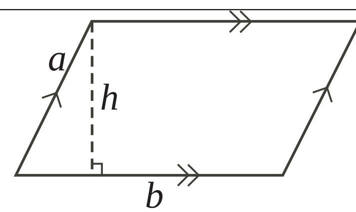
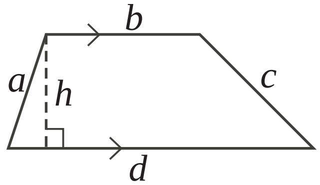
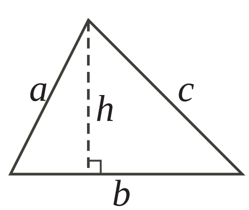
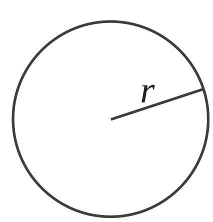


With four pentagons, the sum of the angles in the middle will be over 360° so we cannot create a 3D shape this way.

What about hexagonal faces?
An interior angle of a regular hexagon is 120°, so the sum of angles in the middle for 3 hexagons would be $3 \times 120° = 360°$, which doesn't work. Any shape with more sides than a hexagon also wouldn't work, since its interior angles would be even bigger.

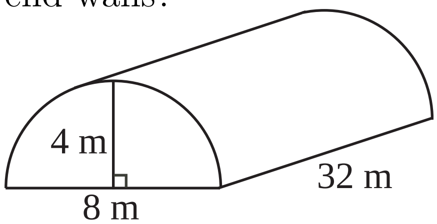
Therefore, we only have 5 platonic solids, created with: three triangles around a vertex, four triangles around a vertex, five triangles around a vertex, three squares around a vertex, or three pentagons around a vertex.

Perimeter and Area
Below is a table of some common 2D figures and the perimeter and area formulas for each, to use while completing the problems.

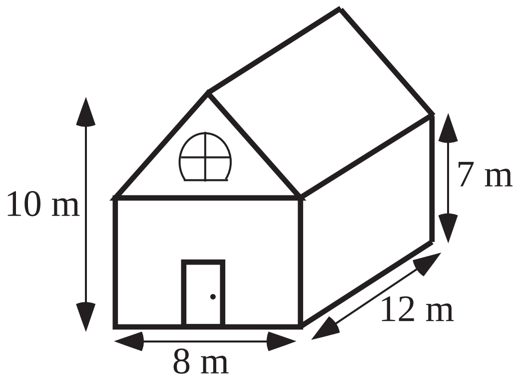
 $P = 4s$ $A = s^2$	 $P = 2(l + w)$ $A = lw$	 $P = 2(a + b)$ $A = bh$
 $P = a + b + c + d$ $A = h\frac{b+d}{2}$	 $P = a + b + c$ $A = \frac{bh}{2}$	 $P = 2\pi r$ $A = \pi r^2$

Problem Set

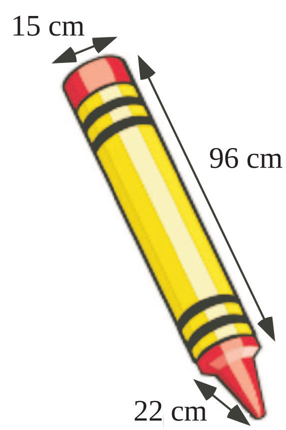
1. If the cross-section of a greenhouse roof is in the shape of a semi-circle with dimensions given below, how much plastic is needed to cover the roof and both end walls?



2. Find the surface area of the house below (not including the floor).



3. Four giant crayons like the one shown at right are being built for a school fair. Each one is painted a different colour: red, yellow, green or blue. What is the total area that must be painted?



4. Which of the 5 platonic solids has the greatest number of edges? Which has the least?

5. A rectangular $4 \times 3 \times 2$ block has its surface painted red, and then is cut into cubes with edge 1 unit. How many cubes have exactly one of their faces painted red?

6. A rectangular prism has a volume of 3696 cm^3. The length is 12 cm and the width is 14 cm. What is its surface area?

7. A soup company is given 4000 cm^2 of aluminum to make a case of 12 soup cans. Their ideal soup can will have a radius of 4 cm and a height of 11 cm.

   (a) Will the company be able to make the 12 soup cans with the amount of aluminum provided?

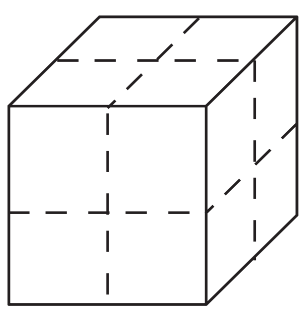
   (b) If not, what is the maximum height each soup can can have in order to make all of the soup cans with only 4000 cm^2 of aluminum (assuming the company still wants the radius of the cans to be 4 cm)?

8. The side length of a square pyramid is 10 cm and the height is 12 cm. The peak of the pyramid lies directly above the centre of the base. What is the surface area of the pyramid?

9. The circumference of the Earth is approximately 40 000 km. What is the surface area of the Earth based on this estimate?

10. A box of crayons that has a volume of 360 cm^3 needs to be packaged. If the length, width and height are whole numbers, what are the dimensions of the box so the least amount of packaging is used?

11. A cube measures 10 cm $\times$ 10 cm $\times$ 10 cm. Three cuts are made parallel to the faces of the cube (as shown below) creating 8 solids which are then separated. What is the increase in the total surface area?



12. **Challenge:** You have models of the five platonic solids. Each model has edge length 1 cm. List the solids from least to greatest surface area.

Answers

1. 144π m$^2 \approx 452.89$ m^2 of plastic
2. 424 m^2
3. 6645π cm$^2 \approx 20875.88$ cm^2 needs to be painted.
4. The icosahedron and dodecahedron each have 30 edges, the greatest number of edges. The tetrahedron has the least number of edges with only 6.
5. 4 cubes have only one face red.
6. The height is 22 cm. The surface area is 1480 cm^2.
7. (a) No, they would need approximately 4524 cm^2 of aluminum.
   (b) The maximum height is approximately 9.26 cm.
8. 360 cm^2
9. Approximately 509 000 000 km^2
10. 6 cm $\times$ 6 cm $\times$ 10 cm
11. The total surface area increases by 600 cm^2 from 600 cm^2 to 1200 cm^2.
12. tetrahedron, octahedron, cube, icosahedron, dodecahedron

Opening Problem

The volume of air is equal to the volume of the cylinder minus the volume of the three tennis balls. The formula for volume of a cylinder is $V = \pi r^2 h$. The height of the cylinder is the height of three tennis balls, and the height of one tennis ball is equal to its diameter, which is twice the radius 2×3 cm $= 6$ cm. Therefore, the height of the cylinder is 3×6 cm $= 18$ cm. The volume of the cylinder is $V = \pi \times 3^2 \times 18 = 162\pi$ cm^3. The formula for volume of a sphere is $V = \frac{4}{3}\pi r^3$, so the volume of the three tennis balls is $V = 3 \times \frac{4}{3}\pi(3)^3 = 108\pi$ cm^3.

The volume of air is therefore $162\pi - 108\pi = 54\pi$ cm$^3 \approx 169.6$ cm^3.