University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
March 3, 2010
3D Geometry I

Opening Problem

Max received a gumball machine for his 12th birthday. Instead of bringing the entire machine for his classmates, he decides to bring 24 gumballs to school - one for each of his classmates. He needs to make a box to stack all of his gumballs to take to school. How many different ways can Max stack his gumballs(one on top of the other and/or side by side) to make a box-like figure?

Solution

Since Max wants to carry them in a box-like figure, we need 3 dimensions: length, width and height. To find the different ways Max can stack his 24 gumballs in these 3 dimensions, we need to find all the factors of 24.
However, we need the factors in groups of 3 instead of pairs of 2.
The different combinations of 3 are: $\{1, 1, 24\}$, $\{1, 2, 12\}$, $\{1, 3, 8\}$, $\{1, 4, 6\}$, $\{2, 2, 6\}$, and $\{2, 3, 4\}$.
The 3 factors in each combination represent the three dimensions of Max's box. So, these 6 combinatios are all the different possibilities of how Max can stack his gumballs in a box-like figure.

Volume of 3D Figures

Definition: The *volume* V of a 3D figure is the amount of space it occupies. The volume of any figure is measured in cubic units (or units cubed).

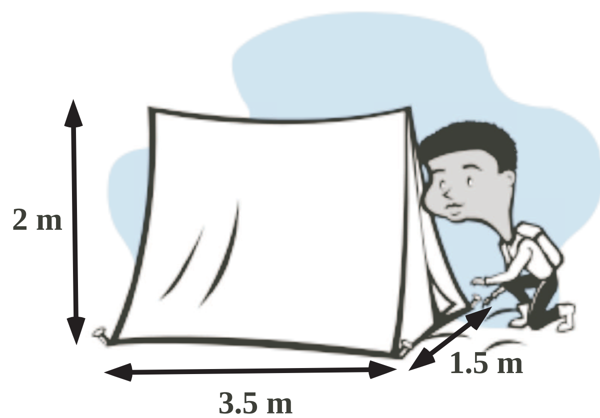
Volume of Prisms

Definition: A *prism* is a 3D figure with two parallel, congruent, polygon-shaped faces that are called *bases*. The remaining faces are either rectangles or parallelograms joining the two bases together. An exception is the *cylinder* which has circular bases and a smooth surface joining them together.

The formula for finding the volume of any prism is:

$$V = \text{area of base} \times \text{height}$$

Example Jerome is building a tent with his father for the first time and wants to fill the entire tent with sleeping bags for fun. The dimensions of his tent are shown below. How many sleeping bags will he be able to fit in his tent if each sleeping bag has a volume of 0.75 m^3?



We can see that the tent is a *triangular prism* since the "bases" are triangles.

To find the volume of this triangular prism tent, first we have to caculate the area of the triangle.

$$\text{Area of Triangle: } \frac{b \times h}{2} = \frac{1.5 \times 2}{2} = 1.5 \text{ m}^2$$

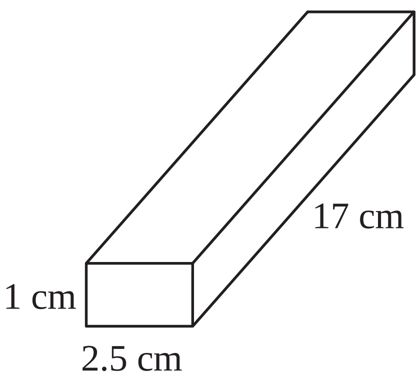
Now, we can multiply the area of the triangle by the height of the tent.

$$V = \text{area of base} \times \text{height} = 1.5 \times 3.5 = 5.25 \text{ m}^3$$

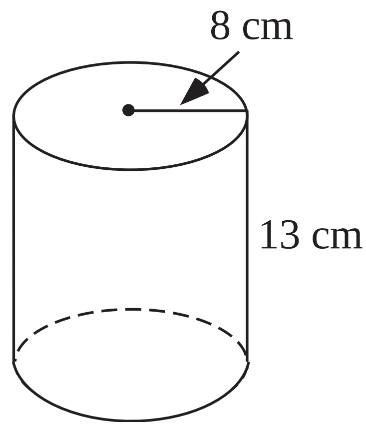
The volume of the tent is 5.25 m^3. Since each sleeping bag is 0.75 m^3 and $5.25 \div 0.75 = 7$, Jerome will be able to fit 7 sleeping bags in his tent.

Exercise 1 Find the volume of the following prisms.

1)



2)



$$\begin{aligned} V &= \text{area of base} \times \text{height} \\ &= (\text{length} \times \text{width}) \times \text{height} \\ &= 1 \times 2.5 \times 17 \\ &= 42.5 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} V &= \text{area of base} \times \text{height} \\ &= \pi r^2 \times \text{height} \\ &= \pi \times 8^2 \times 13 \\ &= 2613.81 \text{ cm}^3 \end{aligned}$$

Volume of Cones & Pyramids

Definition: A *pyramid* is a 3D figure with a polygon base. All other faces are triangles that meet at a common vertex. An exception is the *cone* which has a circular base and a smooth surface that meets at a common vertex.

The volume of pyramids and cones are related to their corresonding prisms. For example, the volume of a triangular prism is related to the volume of a triangular pyramid with the same base and height.

To find the relation, we will do an activity.

1. Create a cylinder and cone that each have the same circular base and the same height. Since we can see that the cone's volume is smaller than the cylinder's, we will find out how many cones it will take to fill the cylinder.

2. Fill the cone with sand, water, or anything else that will accurately measure its volume, then pour it into the cylinder. How many cones do you think will fill the cylinder? 2? 3? 4? more?

3. Continue filling the cone and pouring it into the cylinder until the cylinder is

filled to the top. How many cones did it take to fill the cylinder?

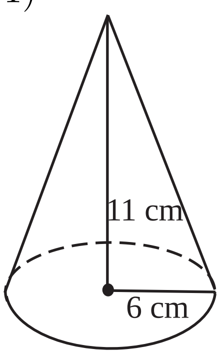
By doing the experiment, we found that it takes 3 cones to fill the cylinder, meaning the volume of the cylinder is 3 times the volume of the cone. This will be the same for any pyramid and the corresponding prism with the same base and height.

So, the volume of a cone or pyramid is:

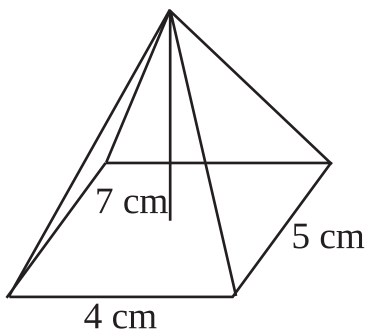
$$V = \frac{1}{3} \times \text{volume of corresponding prism}$$

Exercise 2 Find the volume of the following cone and pyramid.

1)



2)



$$\begin{aligned} V &= \tfrac{1}{3} \times \text{ volume of cylinder} \\ &= \tfrac{1}{3} \times \pi r^2 h \\ &= \tfrac{1}{3} \times \pi \times 6^2 \times 11 \\ &= 414.69 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} V &= \tfrac{1}{3} \times \text{ volume of rectangular prism} \\ &= \tfrac{1}{3} \times lwh \\ &= \tfrac{1}{3} \times 4 \times 5 \times 7 \\ &= 46.67 \text{ cm}^3 \end{aligned}$$

Volume of Spheres

Definition: A *sphere* is a 3D figure that is symmetrical around its center, where all the points on its surface are the same distance from the center point. That distance is called the *radius*.

The formula for finding the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

Example Sarah is planning a birthday party for her friend's 12th birthday. She wants to blow up 12 spherical balloons. If each balloon will have a radius of 14 cm, how much air would Sarah need to fill all 12 balloons?

The question is asking us to find the volume of air needed to fill 12 balloons. So, we can find the volume of 1 balloon then multiply by 12.

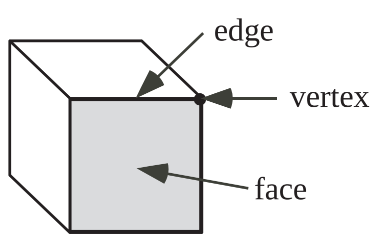
$$\text{Volume of one balloon} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi 14^3 \approx 821\ \text{cm}^3$$

The amount of air needed for 12 balloons is $821\ \text{cm}^3 \times 12 = 9852\ \text{cm}^3$.

Euler's Formula

Definition: A *polyhedron* is a 3D shape with flat, polygon-shaped faces. This means that cones, cylinders and spheres are not polyhedra. A *convex polyhedron* is a polyhedron where you can take any two points on the polyhedron, and the line joining the points will be inside the polyhedron.

Every polyhedron has faces, vertices and edges. A *face* is one of the flat polygon-shaped surfaces. An *edge* is the line segment where two faces meet. A *vertex* is a point where 3 or more edges meet. Below is a diagram displaying a face, an edge and a vertex of a cube.



A famous mathematician, Euler, discovered a relationship involving faces, edges and vertices in the form of an equation. This equation works for all convex polyhedra and most non-convex polyhedra.

Instead of just stating Euler's formula, we will do an activity that will hopefully let you see the relationship. In the table below, fill out the name of the polyhedron you are studying, and the number of faces, edges and vertices that figure has. Continue doing this until your table is full with different polyhedra. Then, using just addition and subtraction, try to figure out the relationship.

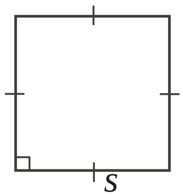
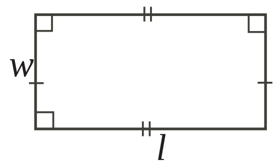
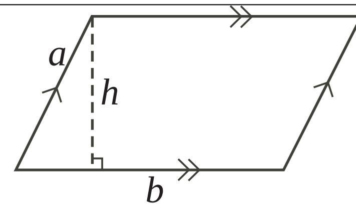
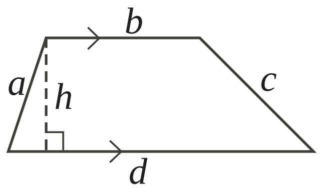
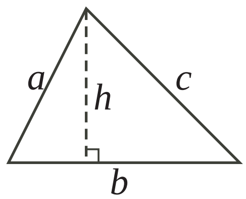
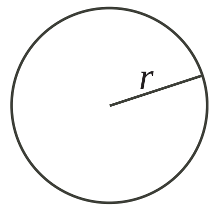
Name of 3D Figure	Faces	Edges	Vertices
Cube	6	12	8
Rectangular Prism	6	12	8
Pentagonal Prism	7	15	10
Triangular Prism	5	9	6
Pyramid	5	8	5
Cuboctahedron	14	24	12
Icosahedron	20	30	12
Tetrahedron	4	6	4
Octahedron	8	12	6
Dodecahedron	12	30	20

Euler's formula is:

$$F + V = E + 2$$

Perimeter and Area

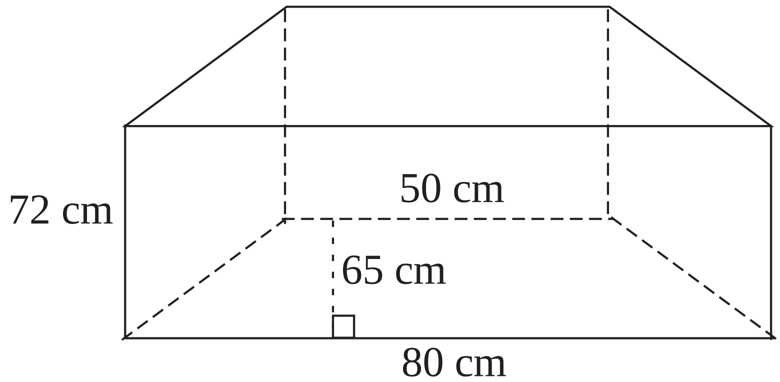
Below is a table of some common 2D figures and the perimeter and area formulas for each, to use while completing the problems.

 $P = 4s$ $A = s^2$	 $P = 2(l + w)$ $A = lw$	 $P = 2(a + b)$ $A = bh$
 $P = a + b + c + d$ $A = h\frac{b+d}{2}$	 $P = a + b + c$ $A = \frac{bh}{2}$	 $P = 2\pi r$ $A = \pi r^2$

Problem Set

1. A cereal company is having a promotion; inside each cereal box there is a pair of dice. The cereal box has dimensions 25 cm by 12 cm by 42 cm. The dice are cubes with side length 3 cm. How much cereal can fit in each cereal box after the dice are placed inside?

2. Mr. Cruise just bought a new fish tank for his daughter and wants to fill it with fresh water. If the fish tank has the dimensions shown at right, how much water will Mr. Cruise need to fill it to the top?



3. A Greek myth says that Atlas the titan was punished and forced to hold the Heavens on his shoulders. If every m^3 of the Heavens weighs 1.2 kg and the Heavens are a sphere with a radius of 2000 m, how much weight is Atlas holding up?

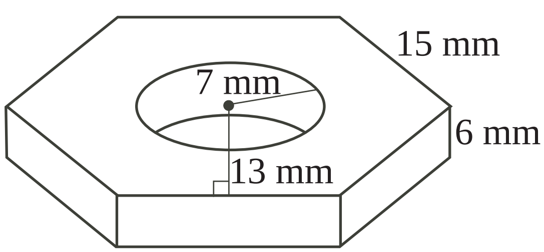
4. An octagonal pyramid has 9 vertices. Using Euler's formula, how many edges does it have?

5. The area of the floor of a rectangular room is 195 m^2. One wall is a rectangle with area 120 m^2 and another wall is a rectangle with area 104 m^2. If the dimensions of the room are all integers, what is the volume of the room?

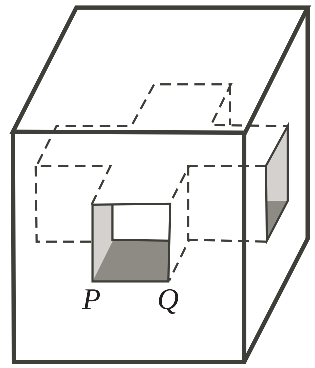
6. Forty-two cubes with 1 cm edges are glued together to form a solid rectangular block. If the perimeter of the base of the block is 18 cm, what is the height?

7. What is the volume of the largest cylinder that will fit inside a cube that has volume 1000 cm^3?

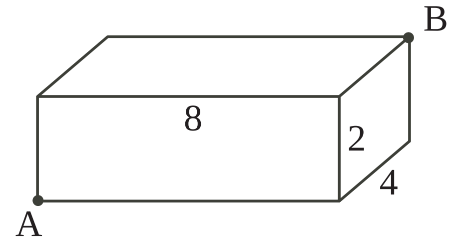
8. What is the volume of the nut shown below? The nut has a regular hexagon base, with a circle cut out. (Hint: A regular hexagon can be split up into 6 identical equilateral triangles.)



9. A cube with side length 10 has two mutually perpendicular square holes of the same size cut out. If $PQ = 6$, what is the volume of the solid once the holes are cut?



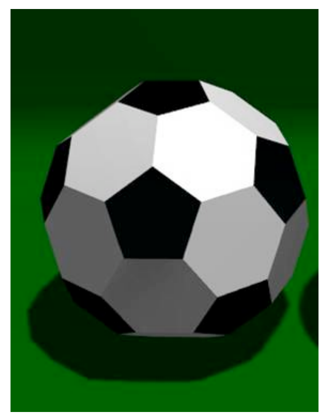
10. An ant wants to walk from point A to point B **on the surface** of the rectangular block shown to the right. What is the shortest path the ant could take?



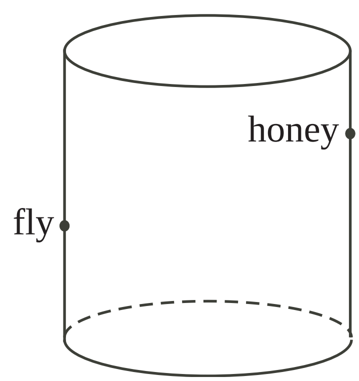
11. The soccer ball shown to the right is a truncated icosahedron. It has 90 edges and 60 vertices.

    (a) How many faces does it have?

    (b) All of its faces are pentagons or hexagons. How many pentagon faces are there? (Hint: If you counted the number of sides of all of the hexagons and pentagons, how many would there be? Every edge joins two sides.)



12. A fly lands on the outside of a cylindrical drinking glass 6 cm from the top. Diametrically opposite the fly and 7 cm from the bottom, but on the inside of the glass, there is a drop of honey. The glass has circumference 24 cm, height 10 cm. Find the shortest path, in cm, that the fly must walk **on the surface** of the glass to reach the honey.



Answers

1. 12 546 cm^3 of cereal
2. 304 200 mL (where 1 cm^3 = 1 mL)
3. About 40 212 385 966 kg
4. 16 edges
5. 1560 m^3, the length of each side is 8 cm, 15 cm, and 13 cm.
6. 3 cm
7. Approx. 785 cm^3
8. Approx. 2586.37 cm^3
9. 496 cm^3
10. 10 cm
11. (a) 32 faces
    (b) 12 pentagon faces
12. 15 cm