



University of Waterloo
Faculty of Mathematics



Centre for Education in
Mathematics and Computing

Junior Math Circles

November 11, 2009

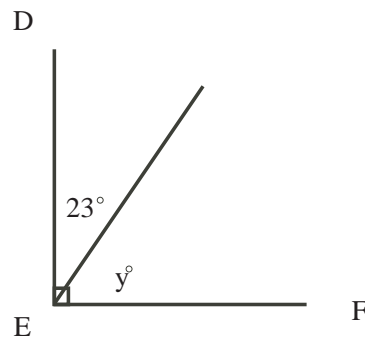
2D Geometry I

What is Geometry?

Geometry is a branch of Mathematics that deals with properties, measurements and relationships of lines, angles, figures and points in space. We will be looking at lines, angles and figures in 2D (two dimensions).

Complementary Angles are angles that add up to 90° .

Practice: If $\angle DEF = 90^\circ$, then what is the value of y ?



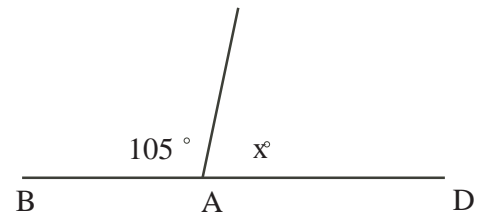
Solution:

$$\begin{aligned} 23^\circ + y^\circ &= 90^\circ \\ y^\circ &= 90^\circ - 23^\circ \\ y^\circ &= 67^\circ \end{aligned}$$

Therefore $y = 67$

Supplementary Angles are angles that add up to 180° .

Practice: If BAD is a straight line, then what is the value of x ?



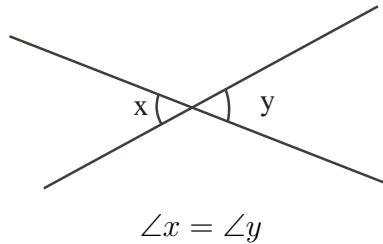
Solution:

$$\begin{aligned} 105^\circ + x^\circ &= 180^\circ \\ x^\circ &= 180^\circ - 105^\circ \\ x^\circ &= 75^\circ \end{aligned}$$

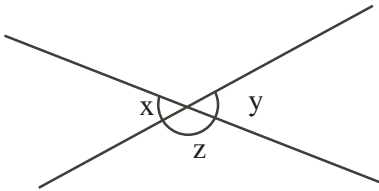
Therefore $x = 75$

Opposite Angle Theorem

The opposite angles formed by two intersecting lines are equal.



Proof:



Let's draw in $\angle z$.

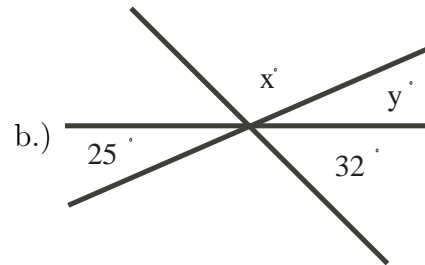
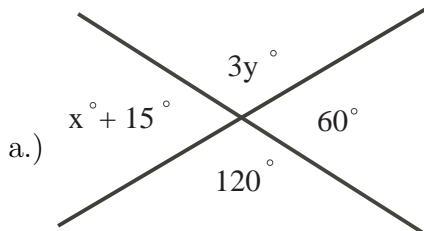
We know that a straight line has an angle of 180° , so we know that:

$$\angle x + \angle z = 180^\circ$$

$$\angle z + \angle y = 180^\circ.$$

For both these equations to be true $\angle x = \angle y$. □

Practice: Determine the values of x and y.



Solutions:

a.) $3y^\circ = 120^\circ$ (opposite angles)

$$\frac{3y^\circ}{3} = \frac{120^\circ}{3}$$

$$y^\circ = 40^\circ$$

$$x^\circ + 15^\circ = 60^\circ \text{ (opposite angles)}$$

$$x^\circ = 60^\circ - 15^\circ$$

$$x^\circ = 45^\circ$$

Therefore $x = 45$ and $y = 40$

b.) $y^\circ = 25^\circ$ (opposite angles)

$$25^\circ + x^\circ + 32^\circ = 180^\circ \text{ (supplementary angles)}$$

$$x^\circ + 57^\circ = 180^\circ$$

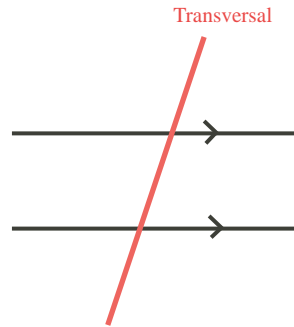
$$x^\circ = 180^\circ - 57^\circ$$

$$x^\circ = 123^\circ$$

Therefore $x = 123$ and $y = 25$

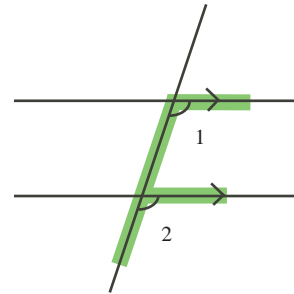
Properties of Parallel Lines and a Transversal

A straight line that intersects two or more lines is called a *transversal*.

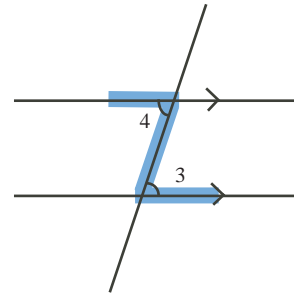


The angles between the parallel lines and a transversal have the following properties:

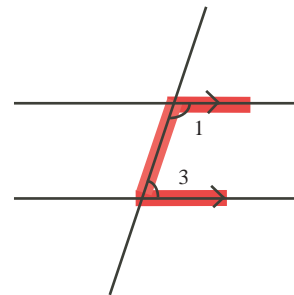
- i) Corresponding angles are equal. $\angle 1 = \angle 2$ (F-pattern)



- ii) Alternate angles are equal. $\angle 3 = \angle 4$ (Z-pattern)

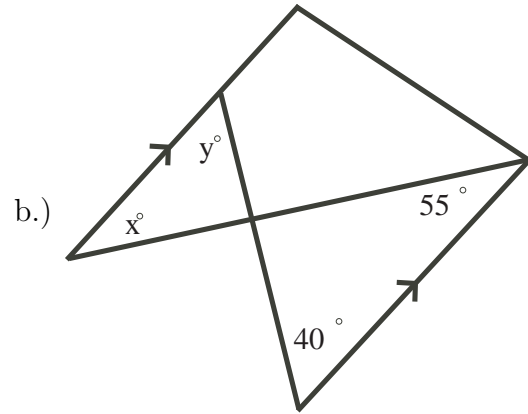
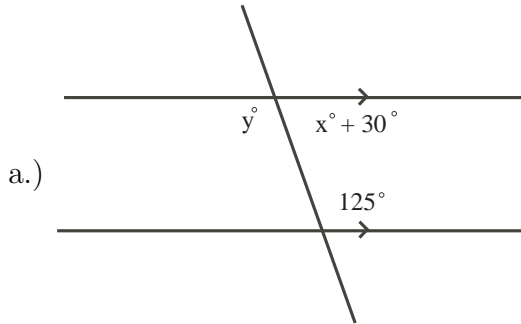


- iii) Interior angles are supplementary. $\angle 1 + \angle 3 = 180^\circ$ (C-pattern)

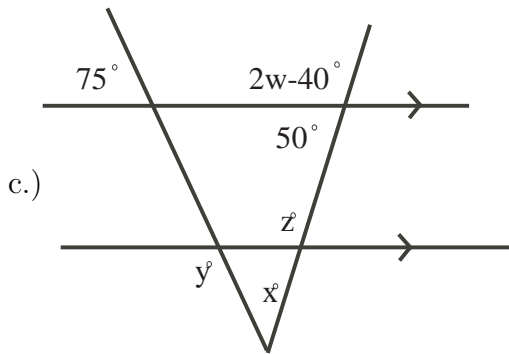


Practice:

Determine the value of x and y .



Determine the value of w , y , x and z .



Solutions:

a.)

$$y^\circ = 125^\circ \text{ (Z-pattern)}$$

$$(x^\circ + 30^\circ) + 125^\circ = 180^\circ \text{ (C-pattern)}$$

$$x^\circ + 155^\circ = 180^\circ$$

$$x^\circ = 180^\circ - 155^\circ$$

$$x^\circ = 25^\circ$$

Therefore $x = 25$ and $y = 125$

b.)

$$y^\circ = 40^\circ \text{ (Z-pattern)}$$

$$x^\circ = 55^\circ \text{ (Z-pattern)}$$

Therefore $x = 55$ and $y = 40$

c.)

$$(2w^\circ - 40^\circ) + 50^\circ = 180^\circ \text{ (supplementary angles)}$$

$$2w^\circ + 10^\circ = 180^\circ$$

$$2w^\circ = 180^\circ - 10^\circ$$

$$2w^\circ = 170^\circ$$

$$\frac{2w^\circ}{2} = \frac{170^\circ}{2}$$

$$w^\circ = 85^\circ$$

$$2w^\circ - 40^\circ = z^\circ \text{ (F-pattern)}$$

$$z^\circ = 2(85^\circ) - 40^\circ$$

$$z^\circ = 130^\circ$$

$$75^\circ + y^\circ = 180^\circ \text{ (supplementary angles)}$$

$$y^\circ = 180^\circ - 75^\circ$$

$$y^\circ = 105^\circ$$

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$$75^\circ + 50^\circ + x^\circ = 180^\circ \text{ (ASTT)}$$

$$125^\circ + x^\circ = 180^\circ$$

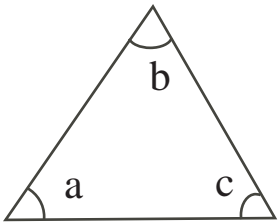
$$x^\circ = 180^\circ - 125^\circ$$

$$x^\circ = 55^\circ$$

Therefore $x=55$, $y=105$, $w=85$ and $z=130$

Angle Sum Triangle Theorem (ASTT)

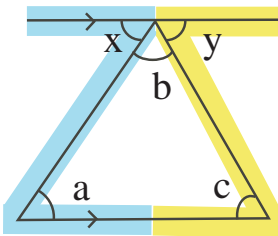
The sum of the interior angles of a triangle is 180° .



$$\angle a + \angle b + \angle c = 180^\circ$$

Proof:

Draw a line at the top of the triangle that is parallel to the base of the triangle.



We know that a straight line has an angle of 180° .

Then we know $\angle x + \angle b + \angle y = 180^\circ$.

In the blue and yellow outlines we can see the Z-pattern from the parallel line properties, showing that $\angle x = \angle a$ and $\angle y = \angle c$.

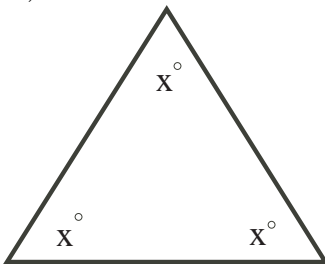
Now if we substitute $\angle a$ for $\angle x$ and $\angle c$ for $\angle y$ into our equation $\angle x + \angle b + \angle y = 180^\circ$ we get,

$$\angle a + \angle b + \angle c = 180^\circ$$

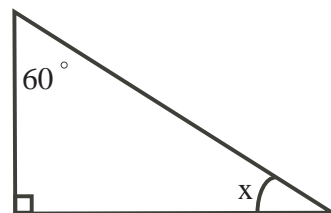
□

Practice:

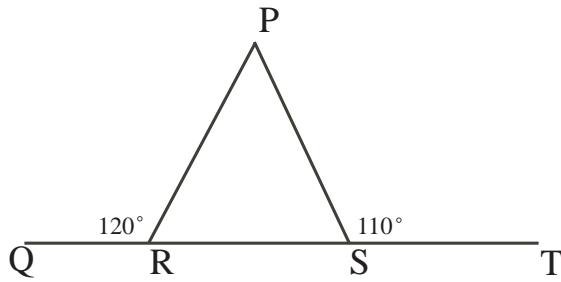
a.) Determine the value of x .



b.) Determine the measure of $\angle x$.



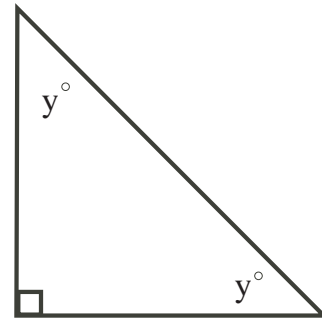
c.) Determine the measure of $\angle RPS$.



Solutions:

$$\begin{aligned}
 \text{a.)} \quad & x^\circ + x^\circ + x^\circ = 180^\circ \text{ (ASTT)} \\
 & 3x^\circ = 180^\circ \\
 & \frac{3x^\circ}{3} = \frac{180^\circ}{3} \\
 & x^\circ = 60^\circ \\
 & \text{Therefore } x = 60
 \end{aligned}$$

d.) Determine the value of y .



$$\begin{aligned}
 \text{b.)} \quad & y^\circ + 60^\circ + 90^\circ = 180^\circ \text{ (ASTT)} \\
 & y^\circ + 150^\circ = 180^\circ \\
 & y^\circ = 180^\circ - 150^\circ \\
 & y^\circ = 30^\circ \\
 & \text{Therefore } y = 30
 \end{aligned}$$

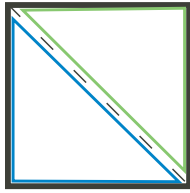
$$\begin{aligned}
 \text{c.)} \quad & \angle PRS + 120^\circ = 180^\circ \text{ (supplementary angles)} \\
 & \angle PRS = 180^\circ - 120^\circ \\
 & \angle PRS = 60^\circ \\
 & \angle RSP + 110^\circ = 180^\circ \text{ (supplementary angles)} \\
 & \angle RSP = 180^\circ - 110^\circ \\
 & \angle RSP = 70^\circ
 \end{aligned}$$

$$\begin{aligned}
 & \angle RSP + \angle RPS + \angle PRS = 180^\circ \text{ (ASTT)} \\
 & 60^\circ + \angle RPS + 70^\circ = 180^\circ \\
 & \angle RPS + 130^\circ = 180^\circ \\
 & \angle RPS = 180^\circ - 130^\circ \\
 & \angle RPS = 50^\circ \\
 & \text{Therefore } \angle RPS = 50^\circ
 \end{aligned}$$

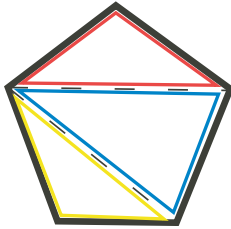
$$\begin{aligned}
 \text{d.)} \quad & y^\circ + y^\circ + 90^\circ = 180^\circ \text{ (ASTT)} \\
 & 2y^\circ = 180^\circ - 90^\circ \\
 & \frac{2y^\circ}{2} = \frac{90^\circ}{2} \\
 & y^\circ = 45^\circ \\
 & \text{Therefore } y = 45
 \end{aligned}$$

Sum of Interior Angles of ANY Polygon

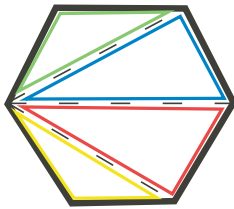
What is the sum of the interior angles of a quadrilateral, pentagon, hexagon or any other n -sided polygon?



With a quadrilateral we can draw a diagonal line dividing the quadrilateral into two triangles. We know the sum of the interior angles of a triangle is 180° , so the sum of the interior angles of a quadrilateral is $2(180^\circ) = 360^\circ$.



With a pentagon we can draw lines dividing the polygon into three triangles, so the sum of the interior angles of a four-sided polygon is $3(180^\circ) = 540^\circ$.

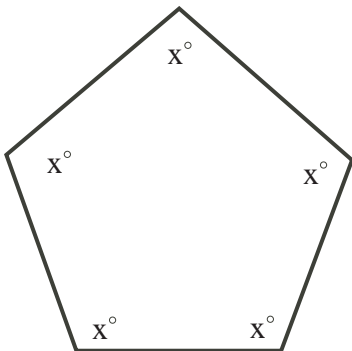


With a hexagon we can draw lines dividing the polygon into 4 triangles, so the sum of the interior angles of a four-sided polygon is $4(180^\circ) = 720^\circ$.

Can you see the pattern? Every side we add to the polygon we increase the sum of the interior angles by 180° . Also notice how the number of triangles inside the shape is two less than the number of sides.

The general formula for the sum of the interior angles of a n -sided polygon is $180^\circ(n - 2)$.

Practice: Find the value of x in the regular pentagon below. Note that x° is the measure of each interior angle.



Solution:

A pentagon has 5 sides.

$$\text{Sum of interior angles} = (5 - 2)(180^\circ)$$

$$x^\circ + x^\circ + x^\circ + x^\circ + x^\circ = 3(180^\circ)$$

$$5x^\circ = 540^\circ$$

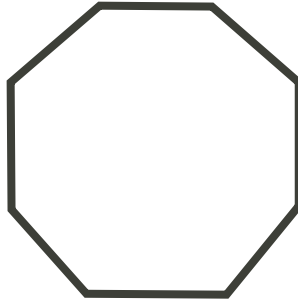
$$\frac{5x^\circ}{5} = \frac{540^\circ}{5}$$

$$x^\circ = 108^\circ$$

Therefore $x=108$

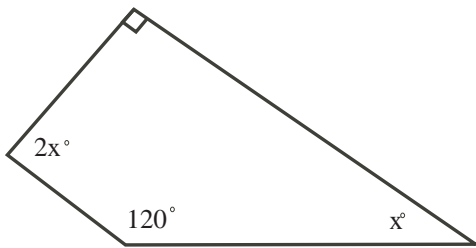
Problem Set

1. Find the measure of each interior angle of the regular octagon below.

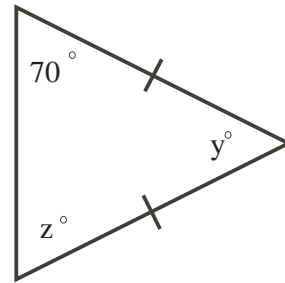


2. In the diagrams below determine the value of each variable.

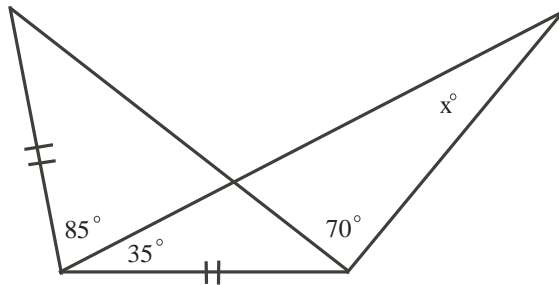
a.)



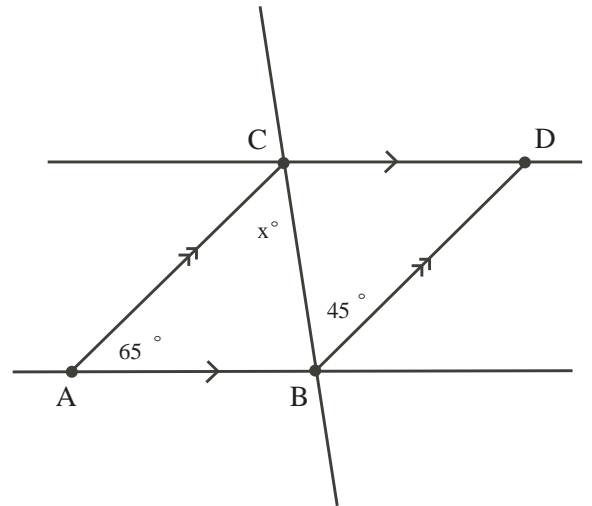
b.)

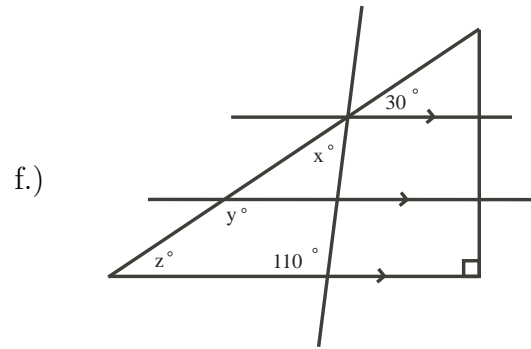
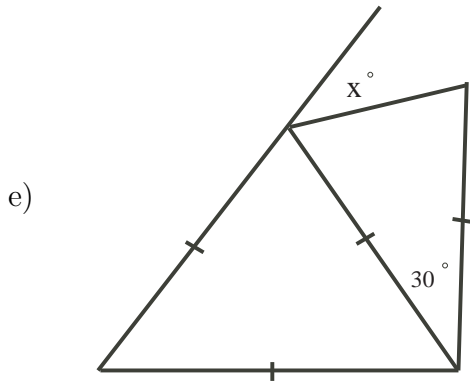


c.)

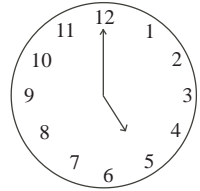


d.)

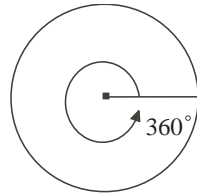




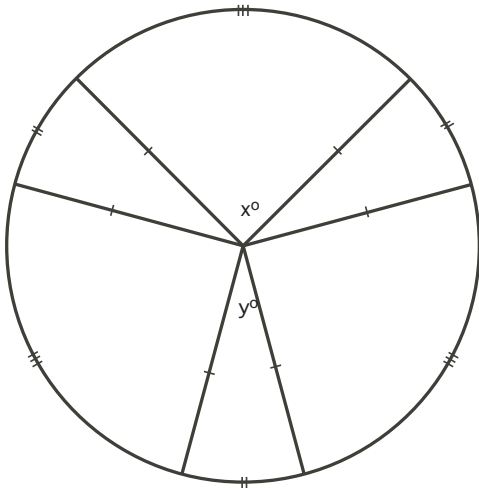
3. What is the angle between the the hour and minute hand on a clock at 5:00pm?



NOTE: Next week we will discuss more about circles, but for now we are going to use that fact that sum of the angles at the centre of a circle is 360° .

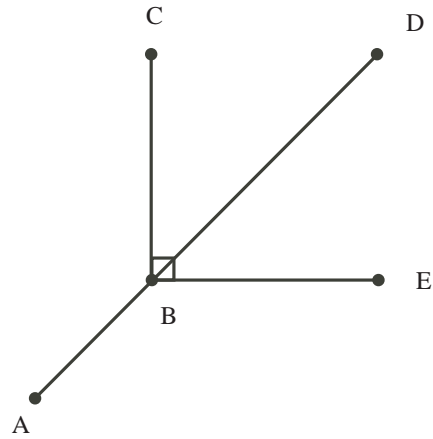


4. In the diagram, what is the value of $x+y$?

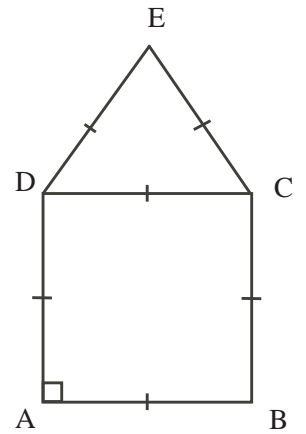


5. What is the angle between the hour and minute hand on a clock at 8:00pm?

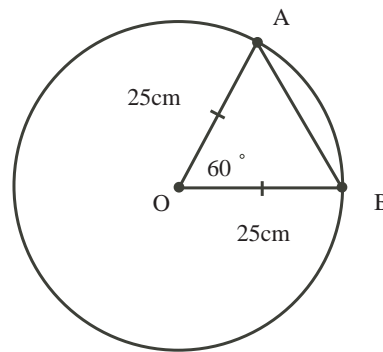
6. Line AD bisects $\angle CBE$. What is the size of $\angle ABE$?



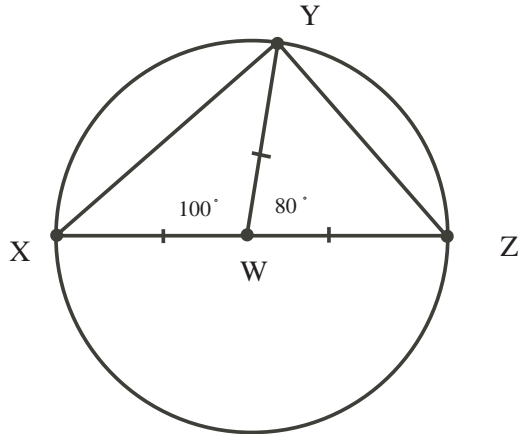
7. $\triangle DEC$ is an equilateral triangle and ABCD is a square. What is $\angle ADE$?



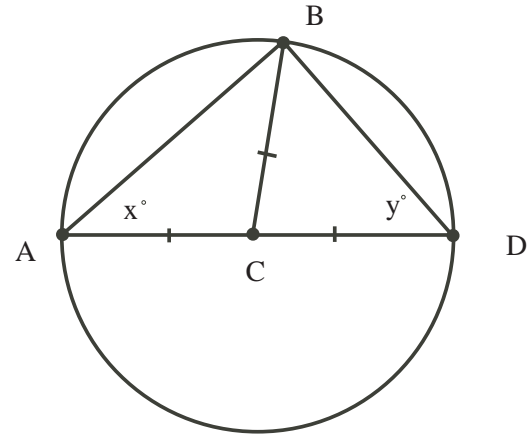
8. What is the length of line segment AB?



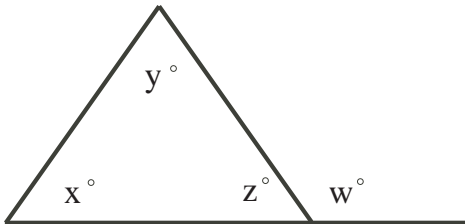
9. a.) Show that $\angle XYZ = 90^\circ$.



b.) Show that $\angle ABD$ is a right-angle no matter what values x and y take as long as x and y are greater than zero, but less than ninety.



10. Show that $x^\circ + y^\circ = w^\circ$.



Answers:

1. $\angle y = 135^\circ$

2. a.) $x = 50$

b.) $z = 70$ $y = 40$

c.) $x = 45$

d.) $x = 45$

e.) $x = 45$

f.) $y = 150$ $z = 30$ $x = 40$

3. 150°

4. $x + y = 120$

5. 240°

6. $\angle ABE = 135^\circ$

7. $\angle ADE = 150^\circ$

8. $|AB| = 25 \text{ cm}$