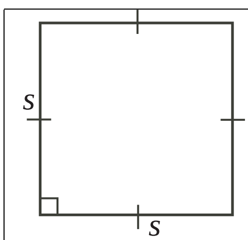
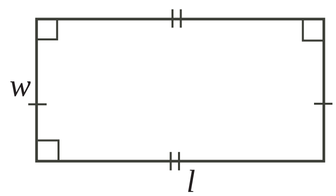
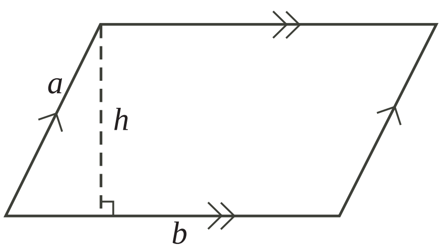
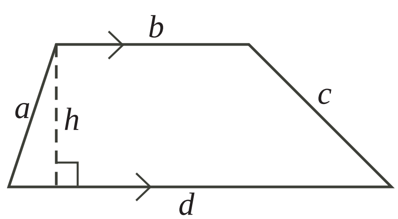
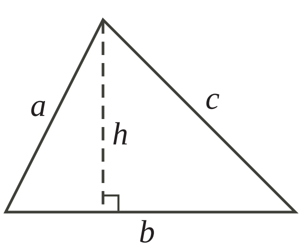
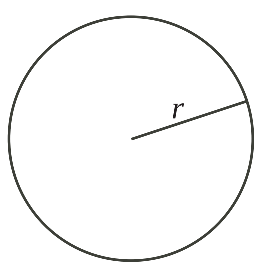
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

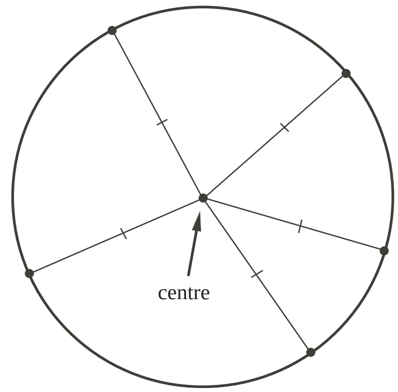
Junior Math Circles
November 18, 2009
2D Geometry II

Two-dimensional shapes have a perimeter and an area.
Perimeter is the length of the outline of a shape.
Area is the surface the shape covers and is measured in units squared (unit^2).
Examples: m^2 or cm^2.

The table below gives formulas for the perimeter P and area A of some common 2-D figures.

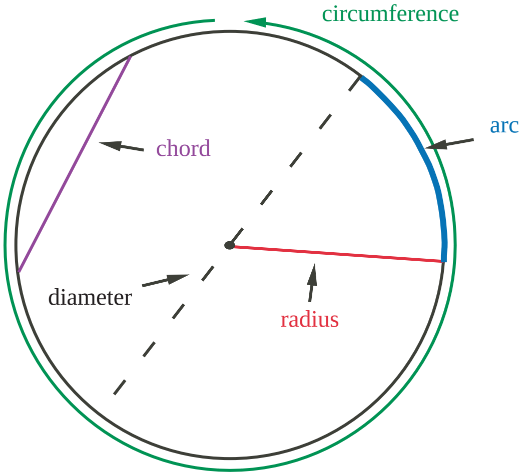
 $P = 4s$ $A = s^2$	 $P = 2(l + w)$ $A = lw$
 $P = 2(a + b)$ $A = bh$	 $P = a + b + c + d$ $A = h\frac{b+d}{2}$
 $P = a + b + c$ $A = \frac{bh}{2}$	 $P = 2\pi r$ $A = \pi r^2$

The Circle



A circle is a figure with the property that every point along its edge is **the same** distance from the centre of the circle.

The perimeter of a circle is called the **circumference**.



The distance from the centre of the circle to any point on the circle's edge is called the **radius**.

A line segment that touches two points on a circle and goes through the centre of the circle is called the **diameter**.

The length of the **diameter** is twice the length of the **radius**.

There is a important relationship between the circumference and diameter of a circle.

$$\frac{\text{Circumference}}{\text{Diameter}} = \frac{C}{d} = \mathbf{3.1415926\ldots}$$

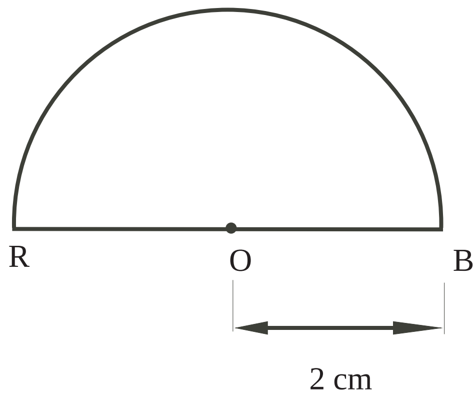
This is true for all circles. This fixed value is called **pi**. It is represented with its lowercase greek letter π.

The equation above can be rearrange to give us a formula for finding the circumference of a circle.

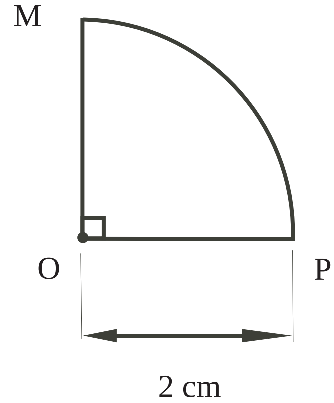
$$\frac{C}{d} = \pi$$
$$\frac{C}{d} \times d = \pi \times d$$
$$C = \pi d$$
$$C = \pi r$$

Exercise 1: Find the perimeter of each figure below. The O is the centre of each of the modified circles below.

1.)



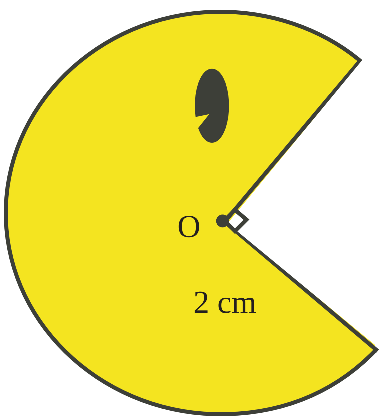
2.)



Length of curve $= \frac{1}{2}(2\pi r)$
$= 2\pi$ cm
Length of lines $= 4$ cm
Perimeter $= 4 + 2\pi$ cm

Length of curve $= \frac{1}{4}(2\pi r)$
$= \pi$ cm
Length of line $= 4$ cm
Perimeter $= 4 + \pi$ cm

3.)



Length of curve $= \frac{3}{4}(2\pi r)$
$= 3\pi$ cm
Length of lines $= 4$ cm
Perimeter $= 4 + 3\pi$ cm

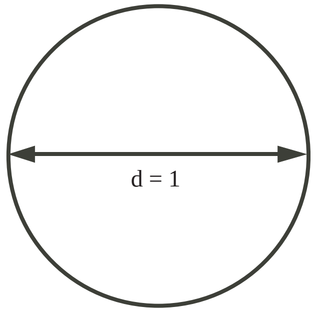
Finding the value for π

People have been trying to compute the value of pi for thousands of years. The ancient Egyptians even developed an approximation for pi. An Egyptian document dating 1650 B.C gives an approximation for pi to be $4(\frac{8}{9})^2 = 3.1605$.(Groleau)

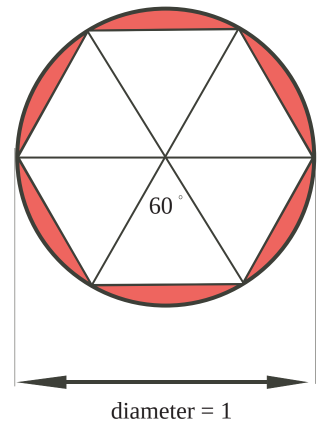
Archimedes developed a theoretical approximation for pi. Archimedes was a greek mathematician, physicist, engineer and inventor that lived from 287 B.C. to 212 B.C. He used *the method of exhaustion* to develop a fairly accurate approximation for pi which is still used today. The method of exhaustion uses polygons to approximate area and perimeter of shapes. We will go over the general idea of how Archimedes approximated π.

First we start with a circle of diameter 1 unit. The units in this instance are not important and could be any standard of measurement, such as m or feet.

$$\begin{aligned} C &= \pi d \\ &= \pi(1) \\ &= \pi \end{aligned}$$

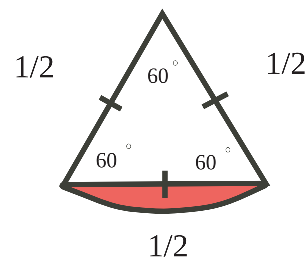


When the diameter is 1 the circumference is π.



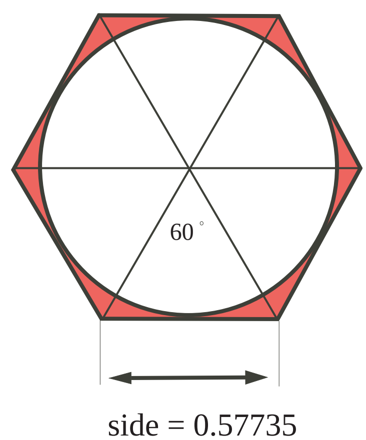
Next we draw a regular hexagon inside the circle with all of the hexagon's vertices touching the edge of the circle. Since the hexagon is contained inside the circle its perimeter will be less than the circumference.

The perimeter of the hexagon is just 6(side length), but what is the side length? We know that the radius of the circle is half the diameter, so we know that the radius is $\frac{1}{2}$.



We know the hexagon inside the circle can be made up of six equilateral triangles with a side length of $\frac{1}{2}$. We know the side length is $\frac{1}{2}$ because two of the sides of the triangle are the length of the radius of the circle.

Then the perimeter of the hexagon inside the circle is $6(\frac{1}{2}) = 3$. The circle's circumference is greater than the perimeter of the hexagon inside the circle, so we know π is greater than 3.

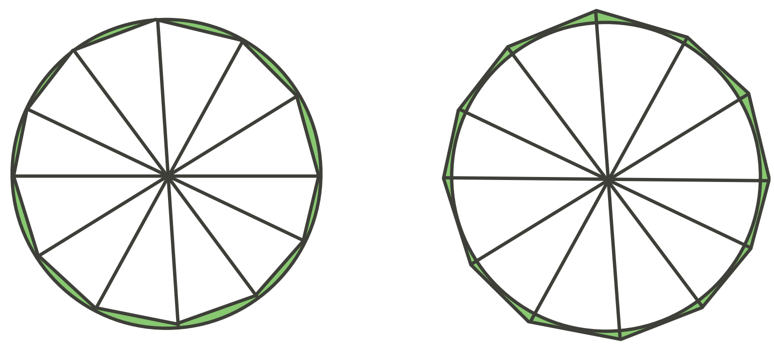


Then we draw a hexagon outside of the circle with the edges of the hexagon touching the circle at one point. To find the side length of this hexagon requires the use of Pythagorean Theorem or Trigonometry. We will learn the Pythagoren Theorem next week, so for now I will tell you the sides' lengths are 0.57735.

The perimeter of the hexagon outside the circle is $6(0.57735) = 3.4641$. The circle's circumference is less than the perimeter of the hexagon outside of the circle, so we know π is less than 3.4641.

Now we know π is greater than 3 and less than 3.464. Using inequalities, $3 < \pi < 3.4641$.

This gives us a large interval for possible values of π. To find a smaller interval for possible value of π we can increase the number of sides of the regular polygon drawn inside and outside the circle. So next we would use a 12-sided polygon and find the perimeter of the polygon drawn inside the circle and the polygon drawn outside the circle.



12-sided polygons

$$3.1058 < \pi < 3.2154$$

The more sides we use the closer the regular polygon's perimeter gets to the value of the circumference of the circle, which in this case is π.

Archimedes used a 96 sided polygon and showed that,

$$\frac{223}{71} < \pi < \frac{22}{7}$$

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

$$3.140845507 < \pi < 3.142857143$$

The value $\frac{22}{7}$ is still used as an approximation for π today.

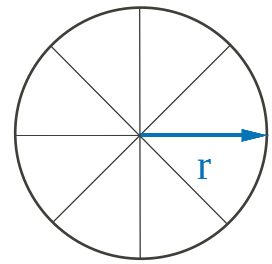
Neat things to note about π :

- in 1610 A.D. Ludolph van Ceulen of the Netherlands used Archimedes' method using polygons with 2^{62} sides. He was able to compute π to 35 decimal places. When he died he had the number engraved on his tombstone.
- Now with the use of computers we π had been approximate to over 2 000 000 decimal places.
- A method to remember a value of π up to 7 decimal places is "**May I have a large container of coffee**?" Replace each word by the number of letters it contains.

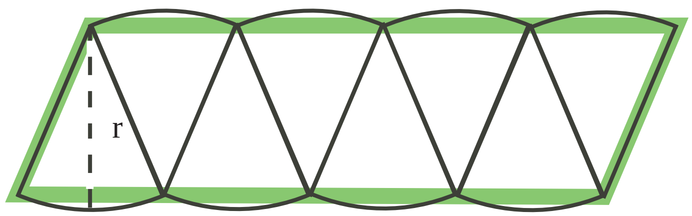
Area of Circle

We know from Page 1 that the formula for area of a circle is $A = \pi r^2$. Now we will show a method of how to develop this formula.

Consider a circle with radius "r". Cut this circle into tiny equal pieces. In the diagram to the right the circle is cut into eight pieces.



Now we rearrange the these pieces into a shape that looks very similar to parallelogram.



Now we know the formula to find the area of a parallelogram is (base)(height). The height is the radius and the base is approximately half the circumference. The more and more equal pieces we cut the circle up into the closer the base becomes to the half the value of the circumference.

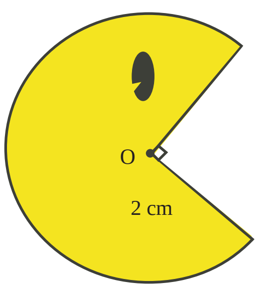
$$\begin{aligned} A =& (base)(height) \\ =& (r)(\tfrac{1}{2}C) \\ =& (r)(\tfrac{1}{2})(2\pi r) \\ =& (r)(\pi r) \\ =& \pi r^2 \end{aligned}$$

$$\begin{aligned} C =& \pi d \\ =& 2\pi r \end{aligned}$$

Therefore the Area of a circle is $A = \pi r^2$.

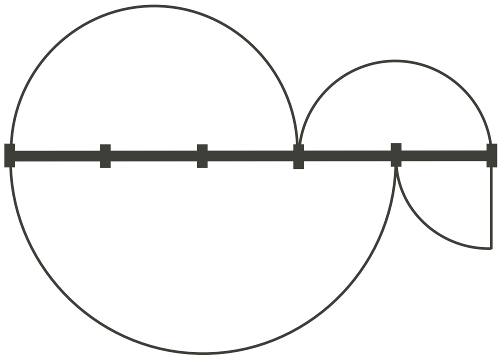
Exercise 2: Find the area of each figure below.

1.)



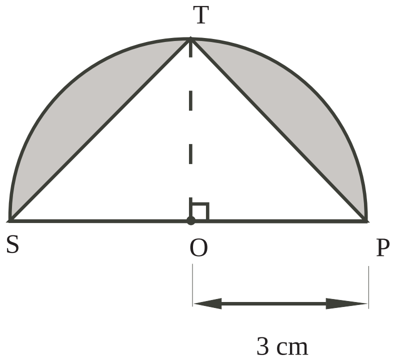
$$\begin{aligned} \text{Area } &= \tfrac{3}{4}(\pi r^2) \\ &= 3\pi \text{ cm}^2 \end{aligned}$$

2.)



Area of top left region $= \frac{1}{2}(\pi r^2) = 1.125\pi$
Area of top right region $= \frac{1}{2}(\pi r^2) = 0.5\pi$
Area of bottom left region $= \frac{1}{2}(\pi r^2) = 2\pi$
Area of bottom right region $= \frac{1}{4}(\pi r^2) = 0.25\pi$
Area $= 3.875\pi$

3.) Find the area of the shaded region.



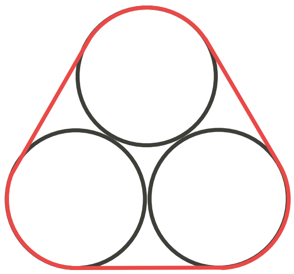
Area of semi-circle $= \frac{1}{2}(\pi r^2) = \frac{9}{2}\pi$ cm^2
Area of triangle $= \frac{1}{2}bh = 9$ cm^2
Area of shaded region $= 9\pi$ cm$^2 - 9$ cm$^2 = 9\pi - 9$ cm^2

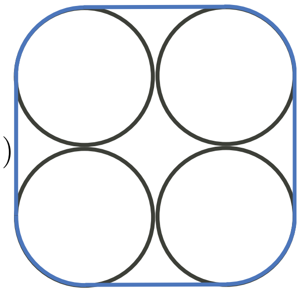
Works cited

Groleau, Rick. Approximating Pi.
21 Oct. 2009 <http://www.pbs.org/wgbh/nova/archimedes/pi.html>

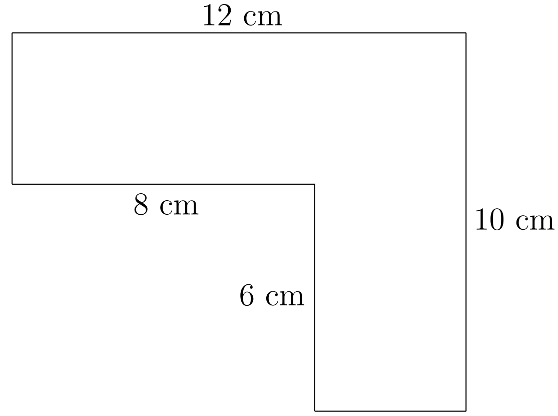
Problem Set

1. Circles are stacked in a triangular and square shape. Each circle has diameter 27.9 cm. Find the length of the rope required to just rap around the outside of each shape as shown in the diagrams below.

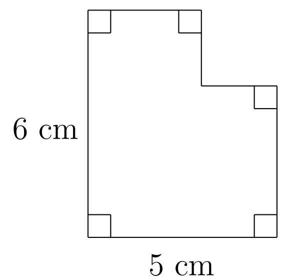
a.) 

b.) 

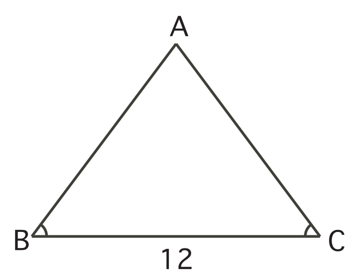
2. What is the perimeter of the figure?



3. (10 G7 2004) What is the perimeter of the figure?



4. (11 G8 2009) The perimeter of $\triangle ABC$ is 32. If $\angle ABC = \angle ACB$ and $BC = 12$, what is the length of AB?

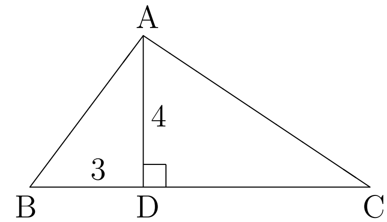


5. What is the radius of a circle having circumference 6π units?

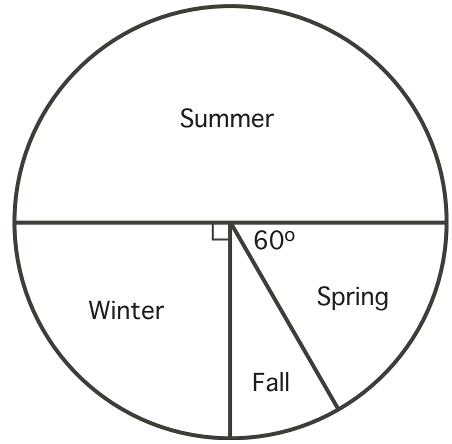
6. The area of a given circle is 9π cm^2. What is the diameter of this circle, in cm?

7. In △ABC, ∠B is 36 degrees larger than ∠A and ∠C is six times ∠A. What is the measure of ∠A?

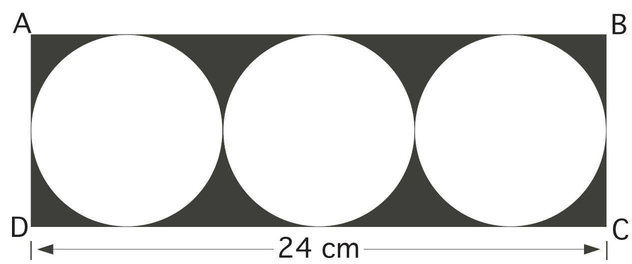
8. (13 G7 2005) In the diagram, the length of DC is twice the length of BD. What is the area of triangle ABC?



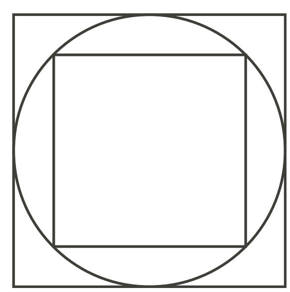
9. (14 G7 2006) In the diagram, O is the centre of the circle, AOB is a diameter, and the circle graph illustrates the favourite season of 600 students. How many of the students surveyed chose Fall as their favourite season?



10. (19 G8 2003) In the diagram, ABCD is a rectangle, and three circles are positioned as shown. What is the area of the shaded region, rounded to the nearest cm^2?



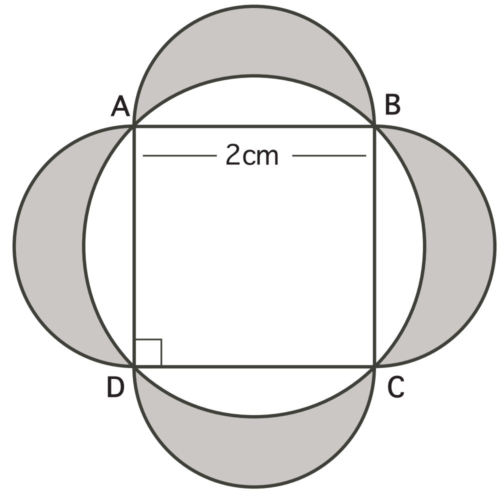
11. (21 G8 2005) In the diagram, a circle is inscribed in a large square and a smaller square is inscribed in the circle. If the area of the large square is 36, what is the area of the smaller square?



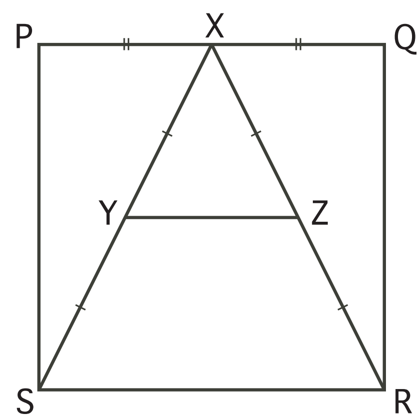
12. (23 G8 2005) A wheel with radius 1 m is rolled in a straight line through one complete revolution on a flat horizontal surface. How many metres did the center of the wheel travel horizontally from its starting location?

13. When the radius of a circle is tripled, how are the area and circumference of the circle affected?

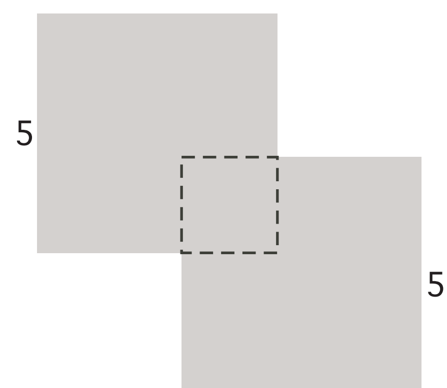
14. Square ABCD with side length 2cm is inscribed in a circle, as shown. Using each side of the square as a diameter, semi-circular arcs are drawn. Find the area of the shaded region outside the circle and inside the semi-circles.



15. (24 G7 2002) PQRS is a square with side length 8. X is the midpoint of side PQ, and Y and Z are the midpoints of XS and XR, respectively, as shown. What is the area of trapezoid YZRS?



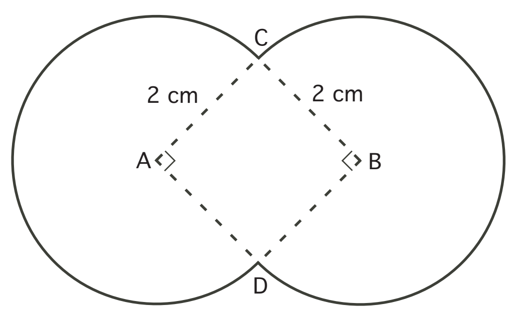
16. (18 G7 2003) Two squares, each with side length 5 cm, overlap as shown. The shape of their overlap is a square, which has an area of 4 cm^2. What is the perimeter, in centimetres, of the shaded figure?



17. Two circles with centres A and B both have a radius of 2 cm. The circles intersect at points C and D such that ∠CAD and ∠CBD are both right angles.

   a) What is the perimeter of the figure?

   b) What is the area of the figure?



Answers

1. a) $73.7 + 27.9\pi$ cm
   b) $101.6 + 27.9\pi$ cm
2. 44 cm
3. 22 cm
4. 10
5. 3 units
6. 6 cm
7. 18 degrees
8. 18
9. 50 students
10. 41 cm^2
11. 18
12. 2π m
13. Area is multiplied by 9 and circumference is tripled.
14. 4 cm^2
15. 24
16. 32 cm
17. a) 6π cm
    b) $4 + 6\pi$ cm^2