University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
November 4, 2009
Algebra II

Arithmetic Trick

If you are multiplying two numbers between 11 and 19 together, there is a trick you can use to make the calculation easier.

Explanation:	Example: 13 and 16
1. Take the units column from one number and add it to the second.	1. $3 + 16 = 19$
2. Multiply your result by 10.	2. $19 \times 10 = 190$
3. Add this result to the product of the two units digits.	3. $190 + 3(6) = 208$

The number you have arrived at is the product of the original two numbers.

The Distributive Law

The Distributive Law states that for any numbers a, b and c:

$$a(b + c) = ab + ac$$

Example:

$$\begin{aligned} 4(5 + 6) &= 4(5) + 4(6) \\ 4(11) &= 20 + 24 \\ 44 &= 44 \end{aligned}$$

The Distributive Law can also be applied with division.

Example:

$$\begin{aligned} \frac{7+5}{4} &= \frac{7}{4} + \frac{5}{4} \\ \frac{12}{4} &= \frac{12}{4} \\ 3 &= 3 \end{aligned}$$

Negatives

In arithmetic, subtracting a positive number or adding the negative of that number performs the same operation. When using the Distributive Law, we treat any subtraction in this way to avoid confusion, putting negative numbers in brackets.

$$\begin{aligned} -6(12-7) &= (-6)(12+(-7)) \\ &= (-6)(12)+(-6)(-7) \\ &= -72+42 \\ &= -30 \end{aligned}$$

Collecting Like Terms

The Distributive Law is more often used to simplify algebraic expressions rather than arithmetic expressions.

Examples:

$$5(a+2) = 5a+10 \qquad\qquad 3(a+(-5)) = 3a-15$$

We have used the Distributive Law to expand expressions but we can also use it to collect like terms.

- A **term** is a part of an expression connected only by multiplication or division.
  Examples: $5a$, 10, $\dfrac{2b}{3}$ are all terms. $5(x+10)$ is not a term.

- **Like terms** contain all of the same variables, but may be multiplied by different numbers.
  Examples: $2x$, $3x$, $12x$ are all like terms. $2a^2$, $4a$, $6b$, 10 are not like terms.

When we collect like terms, we use the Distributive Law, but we are doing the opposite of expanding. Instead, we take out a factor of what each term has in common and write it outside of the terms, putting the affected terms in brackets.

Example:

$$\begin{aligned} (5a+10)+(3a-15) &= 5a+3a+10-15 \\ &= (5+3)a+(10-15) \\ &= 8a-5 \end{aligned}$$

Visualizing the Distributive Law

We can use a variable to illustrate the logic behind the Distributive Law.

$$\begin{aligned}(3+4)a &= 7a \\ &= \underbrace{a+a+a}_{3a}+\underbrace{a+a+a+a}_{4a}\end{aligned}$$

When multiplication is first learned, 4×3 would be explained as $3+3+3+3$. Above the same idea is used. $7a$ can be seen as adding 7 a's together, which can also be written as 3 a's plus 4 a's.

In general, $(b+c)a$ could be seen as adding $(b+c)$ copies of a together, hence why we can write it as $ba+ca$.

Exercise: Remove all brackets from the following expression and collect like terms.

a) $5(x+3)$ $= 5x + 15$

b) $4(3-n)+2(n+4)$ $= 12 - 4n + 2n + 8$
$= -2n + 20$

c) $12x+3y-4x+2y$ $= 8x + 5y$

d) $4(a-b)+3(a+b)$ $= 4a - 4b + 3a + 3b$
$= 7a - b$

Arithmetic Trick

Let a and b be integers between 1 and 9. Then $(10+a)$ and $(10+b)$ are two digit numbers beginning with 1. Lets try our arithmetic trick with these numbers.

Steps: $(10+a)$ and $(10+b)$

1. Take the units column from one number and add it to the second. 1. $a+(10+b)$
2. Multiply your result by 10. 2. $(a+(10+b))(10)$
3. Add this result to the product of the two units digits. 3. $(a+(10+b))(10)+ab$

We will show the arithmetic trick works using the Distributive Law.

$$\begin{aligned}(10+a)(10+b) &= (10+a)(10)+(10+a)b \\ &= 10(10)+10a+10b+ab\end{aligned}$$

Note that multiplying an expression into another expression using the Distributive Law is the same as when it is a number or a variable. Just make sure you multiply each term in the second expression by the entire first expression.

We used the Distributive Law twice to expand this expression. Now we would normally collect like terms, but we are instead going to collect all terms with a factor of 10 in them.

$$\begin{aligned} 10(10) + 10a + 10b + ab &= (10 + a + b)(10) + ab \\ &= (a + (10 + b))(10) + ab \end{aligned}$$

We have now derived our arithmetic trick!

Problems from last week

Below is a selection of solutions to problems from last week. These problems have solutions which make use of the Distributive Law, which is why we have included them here. Enjoy!

Question 4

A box contains 14 disks, each coloured red, blue or green. There are twice as many red disks as green disks, and half as many blue as green. How many disks are green?

Solution

Let R be the number of red disks.
Let B be the number of blue disks.
Let G be the number of green disks.

We then have the following equations:

$$R + B + G = 14 \tag{1}$$

$$R = 2G \tag{2}$$

$$B = \frac{G}{2} \tag{3}$$

Substitute the values of R and B from equations (2) and (3) into equation (1).

$$\begin{aligned} 2G + \frac{G}{2} + G &= 14 \\ (2G + \frac{G}{2} + G)(2) &= 14(2) \\ 4G + G + 2G &= 28 \\ (4 + 1 + 2)G &= 28 \\ \frac{7G}{7} &= \frac{28}{7} \\ G &= 4 \end{aligned}$$

Therefore, 4 of the disks are green.

Question 13

In her backyard garden, Gabriella has 12 tomato plants in a row. As she walks along the row, she notices that each plant in the row has one more tomato than the plant before. If she counts 186 tomatoes in total, how many tomatoes are there on the last plant in the row?

Solution

Let x be the number of tomatoes on the first plant. Then there are:

$x+1$ tomatoes on the second plant,
$x+2$ tomatoes on the third plant,
$x+3$ tomatoes on the fourth plant,
$x+4$ tomatoes on the fifth plant,
$x+5$ tomatoes on the sixth plant,
$x+6$ tomatoes on the seventh plant,
$x+7$ tomatoes on the eighth plant,
$x+8$ tomatoes on the ninth plant,
$x+9$ tomatoes on the tenth plant,
$x+10$ tomatoes on the eleventh plant,
$x+11$ tomatoes on the twelfth plant.

We know that the total number of tomatoes is 186, so:

$$x+(x+1)+(x+2)+(x+3)+(x+4)+(x+5)+(x+6)+(x+7)$$
$$+(x+8)+(x+9)+(x+10)+(x+11)=186$$

The sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$

$\Rightarrow \quad 12x+66=186$

$\Rightarrow \quad 12x=120$

$\Rightarrow \quad x=10$

Therefore, the number of tomatoes on the last plant is $10+11=21$.

Question 12

In her latest game, Mary bowled 199 and this raised her average from 177 to 178. To raise her average to 179 with the next game, what must she bowl?

Solution

Step 1. Find the number of games she originally bowled.

When dealing with averages, we have that

$$\frac{\text{the sum of all scores}}{\text{total number of games}}=\text{average}$$

Let x be the number of games bowled before her latest game.
Let S be the sum of all scores before her latest game. Then,

$$\frac{S}{x} = 177$$
$$\left(\frac{S}{x}\right) x = 177x$$
$$S = 177x$$

We also know that after she bowled 199, her average increased to 178. In other words,

$$\frac{S + 199}{x + 1} = 178$$

Lets substitute $S = 177x$ into our equation and solve for x

$$\left(\frac{177x + 199}{x + 1}\right) (x + 1) = 178(x + 1)$$
$$177x + 199 - 177x = 178x + 178 - 177x$$
$$199 - 178 = x + 178 - 178$$
$$21 = x$$

Step 2 Find what she must bowl to raise her average.

Let y be the score she must bowl to raise her average to 179.

Then:

$$\frac{178(22) + y}{23} = 179$$

Remember that "the sum of all scores = average(number of games)", which is how we got "178(22)".

$$\left(\frac{178(22) + y}{23}\right) (23) = 179(23)$$
$$3916 + y - 3916 = 4117 - 3916$$
$$y = 201$$

Problems

1. Remove all brackets from the following expressions and collect like terms if necessary.

   a) $5(a+2)$ b) $6(f+g)-5(h-g)$

   c) $\dfrac{9+t}{12}$ d) $(a-b)(a+b)$

2. Solve for x in the following expressions.

   a) $5(x+2)=15$ b) $\dfrac{3x+7}{4}=4$

   c) $\dfrac{x+2}{3}=\dfrac{x+3}{2}$

3. (22 G7 2003) Luke has played 20 games and has a 95% winning percentage. Without losing any more games, how many more games in a row must he win to reach exactly a 96% winning percentage?

4. The Math Circles Theatre presented "The Math Before Time" for three days last week. On the second day, 50 more people came to see the play than on the first day. On the third day, the number of people who came to see the play was double the number of people who came on the first day. If the total number of people to attend the play over the three days was 450, how many people came on the first day?

5. (18 G8 2007) The number n is doubled and then had y added to it. The result is then divided by 2 and has the original number n subtracted from it. What is the final result?

6. Using the multiplication trick, complete the following multiplication table.

$\times$	11	12	13	14	15	16	17	18	19
11									
12									
13									
14									
15									
16									
17									
18									
19									

7. (17 G8 2005) If a is an even integer and b is an odd integer, which of the following could represent an odd integer?

   (**A**) ab (**B**) $a + 2b$ (**C**) $2a - 2b$ (**D**) $a + b + 1$ (**E**) $a - b$

8. The average of three numbers is V. If one of the numbers is Z, and one of the numbers is X, what is the third number?

9. (25 G8 2005) A purse contains a collection of quarters, dimes, nickels, and pennies. The average value of the coins in the purse is 17 cents. If a penny is removed from the purse, the average value of the coins becomes 18 cents. How many nickels are in the purse?

10. (25 G8 2004) A large block, which has dimensions n by 11 by 10, is made up of a number of unit cubes and one 2 by 1 by 1 block. There are exactly 2362 positions in which the 2 by 1 by 1 block can be placed. What is the value of n?

11. Remove all brackets from the following expressions and collect like terms if necessary.

    a) $5(a + b + c)$ b) $(r + s)(r + s + t)$

    c) $\dfrac{7 + t + s}{12} + 5t$ d) $(a + b)(a + 2)(b + 1)$

Answers

1. a) $5a + 10$ b) $6f + 11g - 5h$ c) $\frac{3}{4} + \frac{1}{12}t$ d) $a^2 - b^2$
2. a) $x = 1$ b) $x = 3$ c) $x = -5$
3. 5
4. 100
5. $\frac{y}{2}$
6.

×	11	12	13	14	15	16	17	18	19
11	121	132	143	154	165	176	187	198	209
12	132	144	156	168	180	192	204	216	228
13	143	156	169	182	195	208	221	234	247
14	154	168	182	196	210	224	238	252	266
15	165	180	195	210	225	240	255	270	285
16	176	192	208	224	240	256	272	288	304
17	187	204	221	238	255	272	289	306	323
18	198	216	234	252	270	288	306	324	342
19	209	228	247	266	285	304	323	342	361

7. (**E**)
8. 3V − Z − X
9. 2
10. 8
11. a) $5a + 5b + 5c$ b) $r^2 + 2rs + rt + s^2 + st$
    c) $\frac{7}{12} + \frac{61}{12}t + \frac{1}{12}s$ d) $a^2b + ab^2 + a^2 + 3ab + 2b^2 + 2a + 2b$