University of Waterloo
Faculty of Mathematics

Centre for Education in
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Junior Math Circles
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Sets

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Sets

A set is a collection of distinct objects. Any group of objects can be consider a set. The following are examples of different sets:

- {the members of a group for a school project}
- {your favourite socks}
- {the letters of the alphabet}0

Items which belong to a set are called **elements** of a set, and are written inside of curly brackets that denote the set. The elements of a set can be listed descriptively as written above, or can be listed individually like below:

- {Jill, Jack, John}
- {Adidas, Nike, Champs}
- {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

It is important to note that the order in which elements are listed in the set does not matter. An element simply belongs to a set or it does not. For example, the set {1, 3} is the same as {3, 1}.

Quite often, we will represent a set with a capital letter so that we do not have to write out the entire set. We just need to show which letter is associated with which set, as shown below:

- G = {Jill, Jack, John}
- S = {Adidas, Nike, Champs}
- A = {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

In mathematics, we are often concerned with the relationship between sets, such as elements which are shared between sets and the number of objects which are contained in multiple sets.

Unions

Mike is putting together a badminton team, and he wishes to find someone in his class who would be interested in being his partner. He decides he will try asking those who play tennis or baseball.

Let T = {tennis players}
Let B = {baseball players}

The set of people who play tennis *or* baseball is the **union** of the set of tennis players and the set of baseball players.

It is written as $T \cup B$.

Mike soon learns that 3 people in his class have played tennis and 4 people have played baseball. How many people can Mike ask to be his partner?

Well the first inclination may be to say 7, since $3 + 4 = 7$, but first Mike decides to see who is in each group exactly. He finds the following:

T = {Janet, Joshua, Jamal}
B = {Janet, Jamal, Jordan, Jessica}

Therefore, the people who play tennis or baseball are:

$T \cup B$ = {Janet, Joshua, Jamal, Jordan, Jessica}

So there are 5 people that Mike could ask to be his partner.

Why are there only 5 people? This is because we did not take into account that some people might play both tennis and baseball. Since these people are in both sets, we counted them twice when we simply added the number in one set to the number in another set.

Its very important that when you are counting the number of elements in a union of sets that you make sure to count any elements that are in both sets only once.

Exercise 1. Write out the elements in each of the following unions:

1) N = {1, 3, 5, 7, 9}
D = {2, 4, 6, 8, 10}
$N \cup D$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

2) V = {a, e, i, o, u}
W = {v, o, w, e, l, s}
$V \cup W$ = {a, e, i, l, o, s, u, v, w}

3) E = {blue, red, yellow}
C = {red, orange, yellow, green, blue, purple}
$E \cup C$ = {red, orange, yellow, green, blue, purple} = C

Intersections

Now that Mike know who plays tennis and who plays baseball, he decides to ask those who have played both sports to be his badminton partner first. The set of people who play tennis *and* baseball is the **intersection** of the set of tennis players and the set of baseball players.

It is written as T ∩ B.

From before:
T = {Janet, Joshua, Jamal}
B = {Janet, Jamal, Jordan, Jessica}

After examining the sets, we see that Janet and Jamal play both tennis and baseball. Therefore:

T ∩ B = {Janet, Jamal}.

Exercise 2. Write out the elements in each of the following intersections:

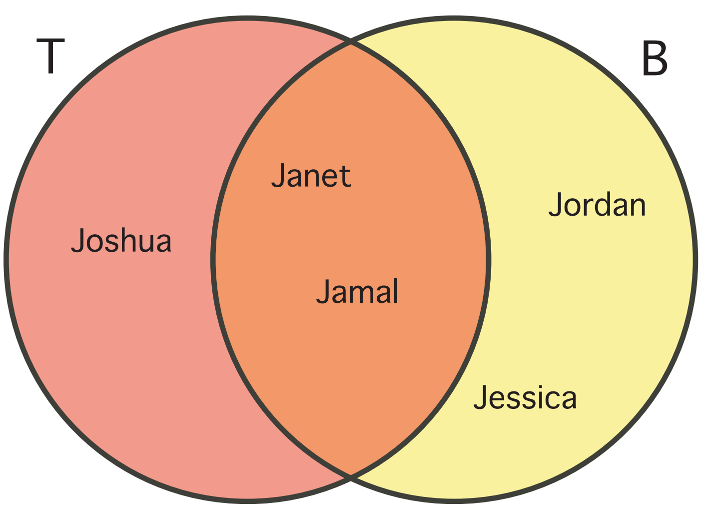
1) N = {1, 2, 3, 4, 5}
D = {2, 4, 6, 8, 10}
N ∩ D = {2, 4}

2) V = {a, e, i, o, u}
W = {s, l, e, w, o, v}
V ∩ W = {e, o}

3) E = {blue, red, yellow}
C = {red, orange, yellow, green, blue, purple}
E ∩ C = {red, yellow, blue} = E

Visualizing Unions and intersections

To visually represent sets, their unions and their intersections, often we will use a Venn Diagram. Below is a Venn Diagram showing the data regarding tennis players and baseball players.



Each of the circles represents a set. The left circle represents T as labelled, and the right circle B. Any of the elements inside the circle belong to the set. Elements that belong to both sets are placed so that they lie in the intersection of T and B. Any element in either circle (including the elements which lie in both) are all part of the union of T and B.

There is an interesting relationship between the number of elements in sets, their unions and their intersections. In our example, we found the following:

$|T|$ = Number of elements in T = 3
$|B|$ = Number of elements in B = 4
$|T \cup B|$ = Number of elements in T $\cup$ B = 5
$|T \cap B|$ = Number of elements in T $\cap$ B = 2

It appears that $|T| + |B| - |T \cap B| = |T \cup B|$. Does this make sense?

Earlier we had discussed that the reason that we could not simply add the number of elements in T and B to find the number of elements in T $\cup$ B was that we were counting some elements twice. But the number of elements counted twice is *exactly* the number of elements in T $\cap$ B. Therefore,

$$|T| + |B| - |T \cap B| = |T \cup B|$$

Complements and Universal Sets

Continuing with the previous example, we now have established there are 5 people Mike could ask to be his partner. But how many of Mike's classmates do not play tennis or baseball?

To decide which people do not belong to the set of tennis or baseball players, we need to first define the set of people we are considering in this situation.

All of the elements which are considered in any particular situation are said to belong to a **universal set**. We will use the letter S to denote the universal set unless otherwise stated.

At the beginning of this example, Mike had decided he would ask someone in his class to be his partner. Therefore the universal set in this case is the set of all Mike's classmates. After examining the attendance sheet, Mike finds that:

S = {Janet, Joshua, Jamal, Jordan, Jessica, Peter, Paula, Patrick, Pricilla, Pam}

Now we consider our original question; How many people do not play tennis or baseball?

The set of people who do *not* play tennis or baseball is called the complement of T $\cup$ B.

It is written as $\overline{\text{T} \cup \text{B}}$.

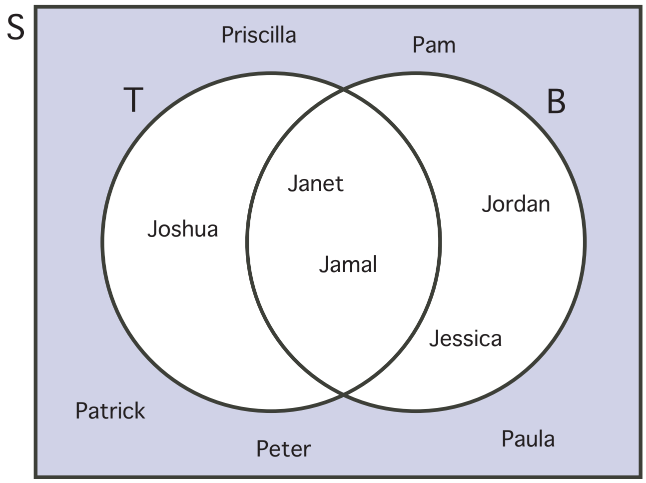
From before:
T $\cup$ B = {Janet, Joshua, Jamal, Jordan, Jessica}

The people who are not in this set are in the complement of this set. Therefore:

$\overline{\text{T} \cup \text{B}}$ = {Peter, Paula, Patrick, Pricilla, Pam}
There are 5 people who do not play tennis or baseball.

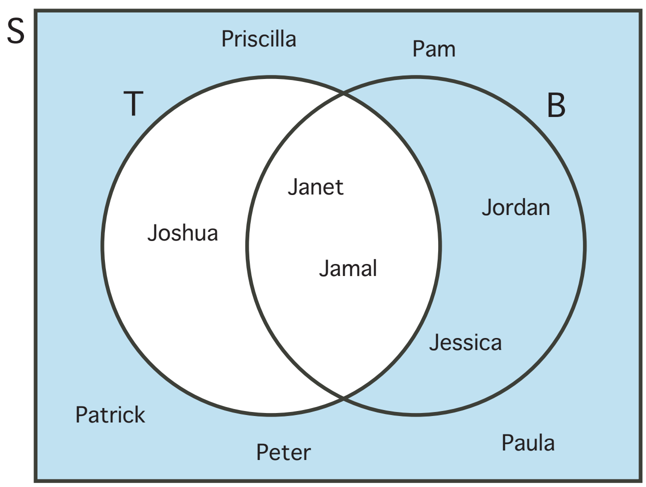
We can also show the above set using a Venn Diagram. The shaded area represents $\overline{\mathrm{T} \cup \mathrm{B}}$.

$\overline{\mathrm{T} \cup \mathrm{B}}$

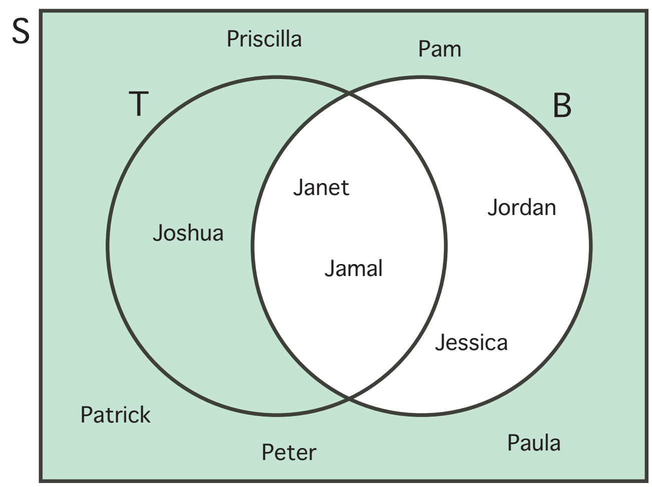


Exercise 3. Shade the area represented by the specified set.

$\overline{\mathrm{T}}$



$\overline{\mathrm{B}}$



Exercise 4. Given the following sets stated below (where S is the universal set), write out the following compliments:

$\mathrm{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$\mathrm{E} = \{2, 4, 6, 8, 10\}$
$\mathrm{O} = \{1, 3, 5, 7, 9\}$
$\mathrm{P} = \{2, 3, 5, 7\}$

1) $\overline{\mathrm{E}} = \{1, 3, 5, 7, 9\}$

2) $\overline{\mathrm{O}} = \{2, 4, 6, 8, 10\}$

3) $\overline{\mathrm{P}} = \{1, 4, 6, 8, 9, 10\}$

4) $\overline{\mathrm{E} \cup \mathrm{P}} = \overline{\{2, 3, 4, 5, 6, 7, 8, 10\}}$
$= \{1, 9\}$

5) $\overline{\mathrm{E} \cap \mathrm{O}} = \overline{\{\}} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

6) $\overline{\mathrm{S}} = \{\}$

The set found in question 6) above is called the **empty set**, since it is a set that contains nothing. It is usually written as $\{\}$ or ϕ.

Subsets and Combinations

You are going to be renovating your room this weekend, and have the option to repaint your room. You have the choice between red, orange, yellow, green, blue and purple. You decide to use green and yellow.

Let C = {chosen colours}
Let S = {colours to choose from}

Since all of the chosen colours belong to the set of colours to choose from, we say that C is a **subset** of S.

It is written as $C \subseteq S$.

Exercise 5. State whether the following statements are true or false.

1) $\{4, 6, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$
True

2) {h, f, g, i} ⊆ {e, g, d, f, i, k, l, h, a}
True

3) $\{3, 5, 7, 9\} \subseteq \{2, 3, 5, 7, 11\}$
False

4) $\{1, 2, 0, 3\} \subseteq \{1203\}$
False

5) {five, four, three} ⊆ {five, four}
False

6) {} ⊆ {Ashley, Michael, Stephen}
True

How many subsets containing two colours can we form?

This sounds very similar to finding the number of permutations. However, we know that the order in which elements are written in a set does not make a difference. A **combination** is an unordered collection of distinct elements.

An easy way to explain the difference between permutations and combinations is this. Permutations are the ways you can *arrange* a certain number of objects; Combinations are the ways you can *choose* a certain number of objects.

First we will find the number of permutations of two colours. We have 6 choices for our first colour and 5 choices for the second colour so:

Number of permutations of 2 colours $= 6 \times 5 = 30$.

But we also know that for any combination of two colours, there are 2! ways to arrange these colours. For example, if we consider the elements of C, we have 2 permutations:

green, yellow
OR
yellow, green

So if we divde the total number of permutations by the number of permutation of two objects, we get that:

$$\text{Number of combinations of 2 colours} = \frac{6 \times 5}{2!} = \frac{30}{2} = 15$$

In general, we can use the following formula:

$$\text{Number of combinations of } k \text{ out of } n \text{ objects} = \frac{n!}{(n-k)! \times k!}$$

Exercise 6. Recall the example where you are renovating your room and have the option to paint your room using any of the colours in S = {red, orange, yellow, green, blue, purple}.

1) How many combinations of 5 colours can you make?

$$\text{Number of combinations of 5 colours} = \frac{6!}{1! \times 5!} = \frac{6 \times \cancel{5 \times 4 \times 3 \times 2 \times 1}}{1 \times \cancel{5 \times 4 \times 3 \times 2 \times 1}} = 6$$

2) How many combinations of 4 colours can you make?

$$\text{Number of combinations of 4 colours} = \frac{6!}{2! \times 4!} = \frac{6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{2 \times 1 \times \cancel{4 \times 3 \times 2 \times 1}} = \frac{30}{2} = 15$$

3) How many combinations of 0 colours can you make? Does this make sense? What are you essentially saying about painting your room if you pick zero colours?

$$\text{Number of combinations of 0 colours} = \frac{\cancel{6!}}{0! \times \cancel{6!}} = 1$$

This makes sense, because not choosing any colours is a choice. What we are saying about painting the room is that we are not going to paint it.

Problems

1. State whether each of the following statements are true or false.

    a) 5 is an element of the set $\{15, 51, 55, 45\}$
    b) $\{\text{vowels}\} \cap \{v, o, w, e, l, s\} = \{\}$
    c) $\{3, 4, 5\} \cup \{7, 6, 5\} = \{3, 4, 5, 7, 6, 5\}$
    d) $\{\text{the integers}\}$ is a set

2. Let P = $\{0, 1, 2, 3, 4, 5\}$, and Q = $\{3, 4, 5, 6, 7, 8\}$. Draw a Venn Diagram of these two sets and enter the integers from 0 to 8 into the correct sections of the diagram.

3. Let A = $\{1, 3, 5, 7, 9\}$, B = $\{2, 4, 6, 8, 10\}$, and C = $\{2, 3, 5, 7, 11\}$. Write out the elements of each of the following sets.

    a) $A \cup C =$
    b) $B \cap C =$
    c) $(A \cap C) \cup B =$

4. An unusual die has its six faces labelled 1, 2, 3, 5, 7, 9. If two such dice are rolled, and the numbers showing on the upper faces are added, what are the number of possible sums?

5. In a class of 30 students, exactly 7 have been to Mexico and exactly 11 have been to England. Of these students, 4 have been to both Mexico and England. How many students in this class have not been to Mexico or England?

6. Draw a Venn Diagram for each part below, shading the area which the set represents.

    a) $\overline{A \cup B}$
    b) $\overline{A} \cup \overline{B}$
    c) $\overline{A} \cap \overline{B}$

    Which of the two sets above are equal?

7. Modify a regular Venn Diagram to show $A \subseteq B$.

8. Given that $A \subseteq B$, complete each of the following:

    a) $A \cap B =$
    b) $A \cup B =$
    c) $\overline{A \cap B} =$
    d) $A \cap \overline{B} =$

9. There are 6 boys and 6 girls interested in becoming a part of the Junior Math Circles Curling Team (a curling team has four team members on it).

   a) How many different teams could be made?

   b) How many different teams have two boys and two girls?

10. Fifty students were surveyed about their participation in hockey and soccer. The results of the survey were:

    33 students played hockey

    24 students played soccer

    8 students played neither hockey nor soccer

    How many of the students surveyed played both hockey and soccer?

11. A class of 30 students was given a short quiz consisting of three questions. The results of the quiz were as follows:

    20 students answered question A correctly

    18 students answered question B correctly

    16 students answered question C correctly

    11 student answered questions A and B correctly

    10 students answered questions A and C correctly

    9 students answered questions B and C correctly

    4 students answered all questions correctly

    Let A = {students who answered question A correctly}
    Let B = {students who answered question B correctly}
    Let C = {students who answered question C correctly}

    Draw a Venn Diagram of these three sets. In each section, write down the number of students which would lie in that particular section of the diagram. (How would you arrange 3 circles so that each overlaps with each other at some point and all three overlap at some point?)

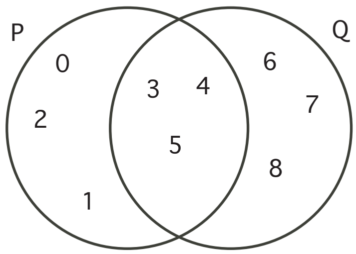
Answers

1. a) False

   b) True

   c) False

   d) True

2. 

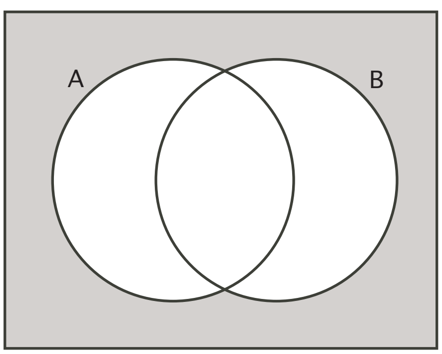
3. a) $\{1, 2, 3, 5, 7, 9, 11\}$

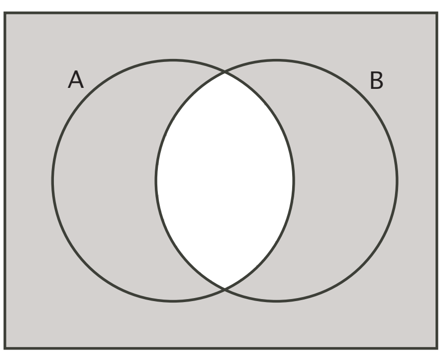
   b) $\{2\}$

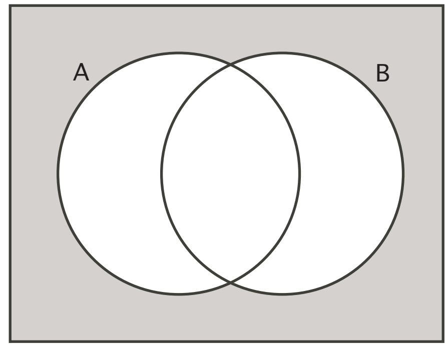
   c) $\{2, 3, 4, 5, 6, 7, 8, 10\}$

4. 14

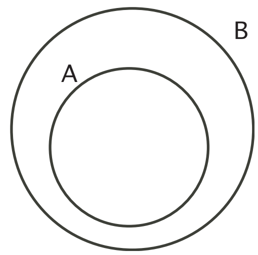
5. 17

6. a) 

   b) 

   c) 

   $\overline{A \cup B} = \overline{A} \cap \overline{B}$

7. 

8. a) A

   b) B

   c) $\overline{A}$

   d) $\{\}$

9. a) 495

   b) 225

10. 15

11.