



University of Waterloo
Faculty of Mathematics



Centre for Education in
Mathematics and Computing

Junior Math Circles

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Probability

Subsets and Combinations

You are going to be renovating your room this weekend, and have the option to repaint your room. You have the choice between red, orange, yellow, green, blue and purple. You decide to use green and yellow.

Let $C = \{\text{chosen colours}\}$

Let $S = \{\text{colours to choose from}\}$

Since all of the chosen colours belong to the set of colours to choose from, we say that C is a **subset** of S .

It is written as $C \subseteq S$.

Exercise 1. State whether the following statements are true or false.

- | | |
|--|---|
| 1) $\{4, 6, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$
True | 2) $\{h, f, g, i\} \subseteq \{e, g, d, f, i, k, l, h, a\}$
True |
| 3) $\{3, 5, 7, 9\} \subseteq \{2, 3, 5, 7, 11\}$
False | 4) $\{1, 2, 0, 3\} \subseteq \{1203\}$
False |
| 5) $\{\text{five, four, three}\} \subseteq \{\text{five, four}\}$
False | 6) $\{\} \subseteq \{\text{Ashley, Michael, Stephen}\}$
True |

How many subsets containing two colours can we form?

This sounds very similar to finding the number of permutations. However, we know that the order in which elements are written in a set does not make a difference. A **combination** is an unordered collection of distinct elements.

An easy way to explain the difference between permutations and combinations is this. Permutations are the ways you can *arrange* a certain number of objects; Combinations are the ways you can *choose* a certain number of objects.

First we will find the number of permutations of two colours. We have 6 choices for our first colour and 5 choices for the second colour so:

Number of permutations of 2 colours = $6 \times 5 = 30$.

But we also know that for any combination of two colours, there are $2!$ ways to arrange these colours. For example, if we consider the elements of C, we have 2 permutations:

green, yellow
OR
yellow, green

So if we divide the total number of permutations by the number of permutation of two objects, we get that:

$$\text{Number of combinations of 2 colours} = \frac{6 \times 5}{2!} = \frac{30}{2} = 15$$

In general, we can use the following formula:

$$\text{Number of combinations of } k \text{ out of } n \text{ objects} = \frac{n!}{(n-k)! \times k!}$$

Exercise 2. Recall the example where you are renovating your room and have the option to paint your room using any of the colours in $S = \{\text{red, orange, yellow, green, blue, purple}\}$.

1) How many combinations of 5 colours can you make?

$$\text{Number of combinations of 5 colours} = \frac{6!}{1! \times 5!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 5 \times 4 \times 3 \times 2 \times 1} = 6$$

2) How many combinations of 4 colours can you make?

$$\text{Number of combinations of 4 colours} = \frac{6!}{2! \times 4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1} = \frac{30}{2} = 15$$

3) How many combinations of 0 colours can you make? Does this make sense? What are you essentially saying about painting your room if you pick zero colours?

$$\text{Number of combinations of 0 colours} = \frac{6!}{0! \times 6!} = 1$$

This makes sense, because not choosing any colours is a choice. What we are saying about painting the room is that we are not going to paint it.

Probability

Probability is the study of how likely an event is to occur. Several different definitions have been created to encompass different situations. The definitions are as follows:

1. The **classical** definition: The probability of an event is

$$\frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$$

provided all outcomes are equally likely. For example, when a die is rolled the probability of getting a 2 is $\frac{1}{6}$ because one of the six faces is 2.

2. The **experimental** definition: The probability of an event is

$$\frac{\text{number of trials in which the event occurred}}{\text{total number of trials in the experiment}}$$

based on data from a long series of repetitions of an experiment or process. For example, this definition could be used to argue that the probability of getting a 2 from a rolled die is $\frac{1}{6}$.

3. The **subjective** definition: The probability of an event is a measure of how sure the person making the statement is that the event will occur. For example, after considering all available data, a weather forecaster might say that the probability of rain today is 30% or 0.3.

In all of these definitions, the probability is a number between 0 and 1, where a probability of 0 would mean that the event never occur, and a probability of 1 would mean that the event always occurs.

Exercise 3. What definition of probability is being used in each of the following situations?

- a) Participating in a raffle
Classical (though Experimental could be argued depending on how the raffle was conducted)
- b) Being 90% sure you passed your math test.
Subjective
- c) Testing several products in a production line to see if they are defective.
Experimental

A **sample space** is a set of distinct outcomes for an experiment or process, with the property that in a single trial, one and only one of these outcomes occurs.

If we used set terminology, the sample space would be called the universal set. We will again use S to represent this set unless otherwise stated.

Example: You roll a six-sided die. If we say that the outcome is the number that is rolled, the sample space for that process is $S = \{1, 2, 3, 4, 5, 6\}$. If we say that the outcome is either an odd number or an even number, the sample space for that process is $S = \{\text{odd, even}\}$. In each of the above sets, one and only one of the listed outcomes will occur.

An **event** is a subset of the sample space. Often we will be interested in one of a number of outcomes occurring, and thus we would calculate the probability of one outcome in a set occurring.

Example. You roll a six-sided die. Given $S = \{1, 2, 3, 4, 5, 6\}$, some events could be:

- {a prime number is rolled} = {2, 3, 5}
- {an even number is rolled} = {2, 4, 6}
- {4, 5, 1}

Could we have the event {a prime number is rolled} defined with specific outcomes if we used the sample space $S = \{\text{odd, even}\}$? This would not be possible, since not all odd numbers are prime, nor are all even numbers.

Quite often we will find it most useful to make the outcomes in the sample space as specific as possible, and then define more compound events.

Each outcome in the sample space has a **probability** associated with it. The notation we use to represent the probability of a specific outcome is $P(\text{outcome})$.

Example. You roll a six-sided die. Given $S = \{1, 2, 3, 4, 5, 6\}$, the probability of the outcomes are:

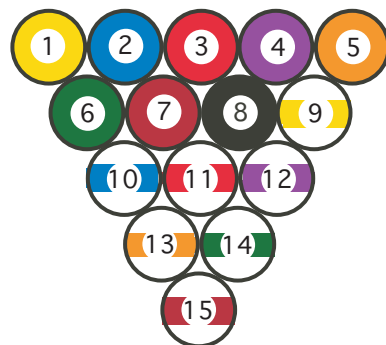
$$P(1) = \frac{1}{6} \quad P(2) = \frac{1}{6} \quad P(3) = \frac{1}{6} \quad P(4) = \frac{1}{6} \quad P(5) = \frac{1}{6} \quad P(6) = \frac{1}{6}$$

When we want to find the probability of an event, we simply add the probability of each outcome in the event.

Example. You roll a six-sided die. Given that each outcome in $S = \{1, 2, 3, 4, 5, 6\}$ has an equal probability, the probability of rolling a prime number:

$$\begin{aligned} P(\text{a prime number is rolled}) &= P(2) + P(3) + P(5) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 4. You have a set of 15 pool balls as shown to the right. If you select one ball at random, calculate the probability of selecting the following:



- $P(\text{solid}) = \frac{8}{15}$
- $P(\text{even}) = \frac{7}{15}$
- $P(\text{solid and even}) = \frac{4}{15}$
- $P(\text{solid or even}) = \frac{11}{15}$
- $P(\text{even and blue}) = \frac{2}{15}$
- $P(\text{striped and black}) = 0$

The exercise above contains several unions and intersections of events. Similarly to last weeks lesson, the number of outcomes in the union of two events is not necessarily the direct sum of the outcomes in each event (parts *a*, *b* and *d* show this)

In fact, the relationship between the number of elements in the union and intersection from last class still holds when we are talking of probabilities. That is to say, for two events *A* and *B*:

$$P(A) + P(B) - P(A \cap B) = P(A \cup B)$$

Exercise 5. Ron, Harry, Neville, and Ginny have a race. Assume all are equally good racers.

- What is the probability that Harry finishes first, Ron second, Neville third and Ginny fourth?
 $P(\text{HRNG}) = \frac{1}{4!} = \frac{1}{24}$
- What is the probability that Harry comes in first place?
 $P(\text{Harry first}) = \frac{1}{4}$
- What is the probability that Harry and Ron finish in consecutive places?
 $P(\text{Harry and Ron beside each other}) = \frac{3! \times 2}{4!} = \frac{1}{2}$

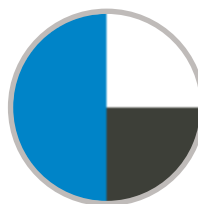
Exercise 6. A bag of marbles contains 4 red marbles, 3 green marbles, and 8 blue marbles.

- Define a sample space for picking 1 marble.
 $S = \{\text{red, green, blue}\}$ since one can pick a red, green or blue marble.
- Define a probability for each outcome in the sample space.
 $P(\text{red}) = \frac{4}{15}$, $P(\text{green}) = \frac{3}{15}$, $P(\text{blue}) = \frac{8}{15}$
- Define a sample space for picking 2 marbles.
 $S = \{\text{rr, rg, rb, gg, gb, bb}\}$ since one can pick two of the same of any colour or some combination of colours.
- Define a probability for each outcome in the sample space.
 $P(\text{rr}) = \frac{\binom{4}{2}}{\binom{15}{2}} = \frac{2}{35}$, $P(\text{rg}) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{15}{2}} = \frac{4}{35}$, $P(\text{rb}) = \frac{32}{105}$, $P(\text{gg}) = \frac{1}{35}$, $P(\text{gb}) = \frac{8}{35}$, $P(\text{bb}) = \frac{4}{15}$

Problems

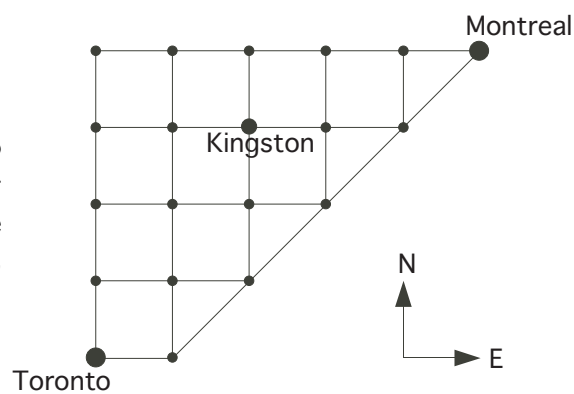
1. A twelve-sided die with sides labelled A, B, C, D, E, F, G, H, I, J, K, L is rolled. What is the probability of rolling a vowel?
2. (Q23 Gr 7 Gauss 2002) Mark has a bag that contains 3 black marbles, 6 gold marbles, 2 purple marbles, and 6 red marbles. Mark adds a number of white marbles to the bag and tells Susan that if she now draws a marble at random from the bag, the probability of it being black or gold is $\frac{3}{7}$. How many white marbles does Mark add to the bag?
3. (Q12 Gr 7 Gauss 2007) The Grade 7 class at Gauss Public School has sold 120 tickets for a lottery. One winning ticket will be drawn. If the probability of one of Mary's tickets being drawn is $\frac{1}{15}$, how many tickets did she buy?

4. A dart is tossed at the board shown here. Assuming the dart is equally likely to hit any spot on the board, what is the probability that it does not hit the black section of the board?



5. (Q9 Gr 8 Gauss 2009) If Jeff picks one letter randomly from the alphabet, what is the probability that the letter is in the word 'probability'?

6. Ash is planning a trip from Toronto to Montreal. He randomly picks a route based on the map to the right, that only ever goes North, East, or North-East. If each route is equally likely to be taken, what is the probability that he does not travel through Kingston?



7. Ashley flips a coin twice. Let $S = \{HH, HT, TH, TT\}$ be the sample space, where HT would represent Heads on the first flip, and Tails on the second flip. List all unique events that could occur (remember that an event is a subset of the sample space). For each, find a way of describing the event in words.

For example, $\{HH, HT\} = \{\text{the first flip lands as Heads}\}$

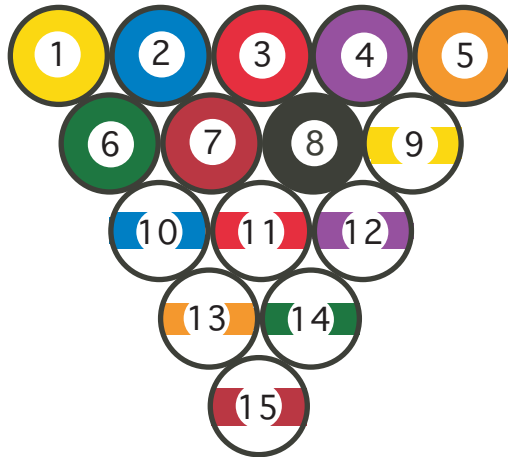
8. (Q20 Gr 8 Gauss 2003) The letters G, A, U, S, and S are written on five tiles, one letter per tile. If Amy selects two tiles at random, what is the probability she gets two S's?
9. Jordan determines that when he spins a coin, it lands on Heads, Tails, or remains on its Side. Determine the sample space if Jordan spins the coin twice.

10. Write out the following combinations in the formation given.

$$\begin{array}{c}
 \binom{0}{0} \\
 \binom{1}{0} \binom{1}{1} \\
 \binom{2}{0} \binom{2}{1} \binom{2}{2} \\
 \binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
 \binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}
 \end{array}$$

What do you notice about the pattern?

11. Bradley asks Julie to grab a file off of his computer, but Julie cannot remember the password. She knows that the password contains the letters of Bradley's name rearranged.
- What is the probability that Bradley's password has two vowels side by side?
 - What is the probability the vowels do not lie side by side?
12. Jeffrey is playing pool with his friend Alexandra. On the first shot (called the break), Jeffrey sinks two balls. It would be to Jeffrey's benefit to sink two balls of the same design, but not to sink the 8 ball. What is the probability that Jeffrey sunk two balls with the same design, not including the 8 ball? (Hint: you may want to consider the chance of sinking two stripes first, then consider sinking two solids other than the 8 ball second, then consider the *union* of the two sets.)



Answers

1. $\frac{1}{4}$

2. 4

3. 8

4. $\frac{3}{4}$

5. $\frac{9}{26}$

6. $\frac{5}{9}$

7. $\{\} = \{\text{none of the outcomes occur}\}$
 $\{\text{HH}\} \quad \{\text{HT}\} \quad \{\text{TH}\} \quad \{\text{TT}\}$
 $\{\text{HH, HT}\} = \{\text{the first flip lands as Heads}\}$
 $\{\text{HH, TH}\} = \{\text{the second flip lands as Heads}\}$
 $\{\text{HH, TT}\} = \{\text{flips land on the same side}\}$
 $\{\text{HT, TH}\} = \{\text{flips land on different side}\}$
 $\{\text{HT, TT}\} = \{\text{the second flip lands as Tails}\}$
 $\{\text{TH, TT}\} = \{\text{the first flip lands as Tails}\}$
 $\{\text{HH, HT, TH}\} = \{\text{not TT}\}$
 $\{\text{HH, HT, TT}\} = \{\text{not TH}\}$
 $\{\text{HH, TH, TT}\} = \{\text{not HT}\}$
 $\{\text{HT, TH, TT}\} = \{\text{not HH}\}$
 $\{\text{HH, HT, TH, TT}\} = \{\text{an outcome occurs}\}$

8. $\frac{1}{10}$

9. $S = \{\text{HH, HT, HS, TH, TT, TS, SH, ST, SS}\}$

$$\begin{array}{c}
 1 \\
 1 \ 1 \\
 10. \quad 1 \ 2 \ 1 \\
 \quad \quad 1 \ 3 \ 3 \ 1 \\
 \quad \quad \quad 1 \ 4 \ 6 \ 4 \ 1
 \end{array}$$

Each number is the sum of the two numbers above it.

11. a) $\frac{2}{7}$

b) $\frac{5}{7}$

12. $\frac{2}{5}$