University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
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Algebra I

Opening Problem

Select any positive integer and perform the following operations.

1. Multiply the number by 4.
2. Add 10
3. Divide by 2
4. Subtract 1
5. Divide by 2 again.
6. Subtract the number that you first started with.

What number will you obtain?

Solution:

If we choose 8, we get:

Step 1. $8 \times 4 = 32$
Step 2. $32 + 10 = 42$
Step 3. $42 \div 2 = 21$
Step 4. $21 - 1 = 20$
Step 5. $20 \div 2 = 10$
Step 6. $10 - 8 = 2$

If we choose 10, we get:

Step 1. $10 \times 4 = 40$
Step 2. $40 + 10 = 50$
Step 3. $50 \div 2 = 25$
Step 4. $25 - 1 = 24$
Step 5. $24 \div 2 = 12$
Step 6. $12 - 10 = 2$

So the final answer in both cases is 2. We can use algebra to show that this is true for any number.

Elementary Algebra

In arithmetic, numbers and operation symbols $(+, -, \times, \div)$ are used to create mathematical expressions and equations.

Examples:

$$2 + 10 \div 5 - 1 \times 4 \qquad\qquad 5 + 7 \times 2 = 19$$

In algebra, numbers and operation symbols are still used to form expressions and equations, but new symbols called **variables** are used as well to represent numbers. The symbol used can be anything, but generally alphabetic letters are used.

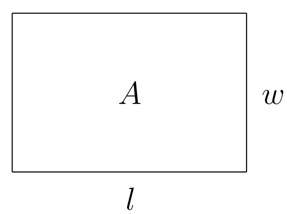
Examples:

$$2 + x \div 5 - 1 \times y \qquad\qquad 5 + 7 \times a = b$$

Variables are useful when we know what an expression or equation should look like, but do not know what the numbers are yet.

For example, the area of a rectangle is equal to the length times the width. If we let A be a variable which represents the area of a rectangle, l be the variable which represents the length of the rectangle and w be a variable that represents the width of the rectangle, then we have:

$$A = l \times w$$



This is a formula for the area of a rectangle. A formula is an algebraic equation which states some sort of rule. In the case above, the area of any rectangle is always equal to the length times the width, though the value of the length and width may change.

Abbreviation

In algebraic equations, it is common practice to abbreviate $\times$ and $\div$ by doing the following:

$$(\text{Expression 1}) \times (\text{Expression 2}) \rightarrow (\text{Expression 1})(\text{Expression 2})$$

Example: $$a \times b \rightarrow ab$$

Two terms written next to one another is often how multiplication is shown in algebra, and any brackets around the expressions must be kept.

$$(\text{Expression 1}) \div (\text{Expression 2}) \rightarrow \frac{\text{Expression 1}}{\text{Expression 2}}$$

Example: $$a \div b \rightarrow \frac{a}{b}$$

Note that the outside brackets can be removed from the two expressions. With a horizontal line, every operation above and below the line must be completed before dividing the numerator from the denominator.

$$(\text{Expression 1}) \times (\text{Expression 1}) \rightarrow (\text{Expression 1})^2$$

Example: $$a \times a \rightarrow a^2$$

The notation you see above is called power notation. Powers have a base, which is what is to be multiplied (above, (Expression 1) is the base), and powers have an exponent, which is how many copies of the base are to be multiplied together.

Exercise 1. Abbreviate the following expressions:

1) $4 \times 4 + n \times n$ $= 4^2 + n^2$

3) $6 \times (2 \div x)$ $= 6\left(\frac{2}{x}\right)$

2) $(a - b) \div (a + b)$ $= \frac{a-b}{a+b}$

4) $x \times x \times x \times x \times x \times x$ $= x^6$

Exercise 2. Expand the following expressions:

1) mn $= m \times n$

3) $a^3 - b^2$ $= a \times a \times a - b \times b$

2) $\frac{p+q}{r}$ $= (p + q) \div r$

4) x^2y^2 $= x \times x \times y \times y$

Substitution

If we are given an expression or equation and specific values for the variables in the formula, we can substitute the values into the expression or equation.

Example: If $x = 4$, and we wish to evaluate $3x + 5$, then

$$\begin{aligned} 3x + 5 &= 3(4) + 5 \\ &= 12 + 5 \\ &= 17 \end{aligned}$$

Exercise 3. Evaluate the following expressions given that $x = 5$ and $y = 4$:

1) y $\quad = 4$

2) $x + y$ $\quad = (5) + (4)$

$\quad = 9$

3) $x^2 + 5y$ $\quad = (5)^2 + 5(4)$

$\quad = 25 + 20$

$\quad = 45$

4) $\dfrac{8x}{y}$ $\quad = \dfrac{8(5)}{(4)}$

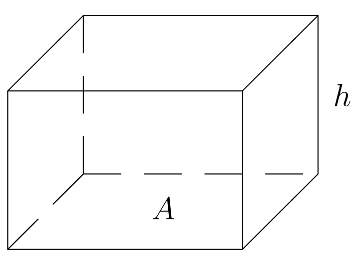
$\quad = \dfrac{40}{4}$

$\quad = 10$

In addition to being able to substitute numbers for variables in expressions, we can also substitute other variables.

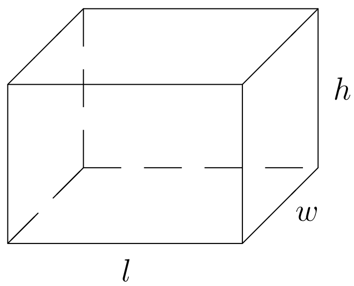
For example, the volume of a rectangular prism is equal to the area of the base times the height. Let V be the volume of the prism, A be the area of the base, and h be the height. Then like with the rectangle before, we can write the following formula:

$$V = A \times h$$



We know from earlier that the formula for area of a rectangle is $A = l \times w$. Therefore, we can substitute $l \times w$ for A in the formula for volume:

$$\begin{aligned} V &= A \times h \\ V &= (l \times w) \times h \\ V &= l \times w \times h \end{aligned}$$



Solving equations

When we are given an equation containing a variable, not often will it be given to us in the form $x =$ "a number", but more likely somewhere embedded in the equation.

Example:

$$\frac{2x+4}{3} = 4$$

To find the value of x, we can solve for x. Solving an equation is the process of performing operations to both sides of an equation until we have the variable we are solving for on one side of the equation by itself. The operations we can perform are addition, subtraction, multiplication and division.

When deciding what operation to do next, try and eliminate a term which affects the whole expression involving x.

Example:

$$\frac{2x+4}{3} = 4$$

=> $$\left(\frac{2x+4}{3}\right) \times 3 = 4 \times 3$$

=> $$2x + 4 - 4 = 12 - 4$$

=> $$\frac{2x}{2} = \frac{8}{2}$$

=> $$x = 4$$

Above, the entire left expression was affected by the division by 3, so we first multiplied by 3 to eliminate the denominator. We then proceeded to subtract 4 to isolate $2x$ by itself, and finally divided by 2 to have an equation with only x on one side.

This process preserves equality because the same operation is always applied to both sides. It is important to note that when multiplying or dividing, the operation must be applied to the entire side of the equation.

Exercise 4. Solve for x in the following equations.

1) $3x + 1 = 7$

$$\Rightarrow \quad 3x + 1 - 1 = 7 - 1$$
$$\Rightarrow \quad \frac{3x}{3} = \frac{6}{3}$$
$$\Rightarrow \quad x = 2$$

2) $y = 4x$

$$\Rightarrow \quad \frac{y}{4} = \frac{4x}{4}$$
$$\Rightarrow \quad \frac{y}{4} = x$$
$$\Rightarrow \quad x = \frac{y}{4}$$

3) $x - 2 = -2$

$$\Rightarrow \quad x - 2 + 2 = -2 + 2$$
$$\Rightarrow \quad x = 0$$

4) $x + 5 = \frac{x-2}{2}$

$$\Rightarrow \quad 2(x+5) = 2\left(\frac{x-2}{2}\right)$$
$$\Rightarrow \quad 2x + 10 - x = x - 2 - x$$
$$\Rightarrow \quad x + 10 - 10 = -2 - 10$$
$$\Rightarrow \quad x = -12$$

Solution (continued):

Lets go back to the opening problem. We will follow the same sequence of steps, but we will use the variable n to represent our number.

Choose a number	Example: Number is 10	Using Algebra: Number is n
1. Multiply by 4.	$10 \times 4 = 40$	$n \times 4 = 4n$
2. Add 10.	$40 + 10 = 50$	$4n + 10$
3. Divide by 2.	$50 \div 2 = 25$	$(4n + 10) \div 2 = 2n + 5$
4. Subtract 1.	$25 - 1 = 24$	$(2n + 5) - 1 = 2n + 4$
5. Divide by 2 again.	$24 \div 2 = 12$	$(2n + 4) \div 2 = n + 2$
6. Subtract starting number.	$12 - 10 = 2$	$(n + 2) - n = 2$
Final Answer	2	2

Algebra allows us to state that no matter what beginning number we select, if we perform these same six steps, our final answer will always be 2.

Problems

1. Determine the value for $x = 4$ and $y = 3$.

a) $4x + y - 7$	b) $6(5) + 2x$	c) $\frac{8y}{3} + 4$
d) $\frac{4y}{6} + 5 - 3x$	e) $\frac{2x}{4y}$	f) $5y + 3x - 2y$

2. Solve for x

a) $x + 5 = 19$	b) $x - 9 = 32$	c) $\frac{2x}{3} - 13 = 17$
d) $\frac{x}{2} = 7$	e) $4x + 5 = 25$	f) $8x = 24$
g) $7x + 12 = 6x$	h) $\frac{2x+2}{2} = 1$	i) $\frac{x+1}{4} = \frac{x-1}{3}$

3. (17 G7 2008) The length of a rectangle is 6 more than twice its width. If the perimeter of the rectangle is 120, what is its width?

4. (14 G7 2003) A box contains 14 disks, each coloured red, blue or green. There are twice as many red disks as green disks, and half as many blue as green. How many disks are green?

5. Try different values with this sequence of operations to find a pattern to your answers. Explain the number trick by showing the steps using algebra.
Pick a number.
   (a) Multiply by 5.
   (b) Add 3.
   (c) Multiply by 2.
   (d) Add 4.
   (e) Divide by 10.
   (f) Subtract 1 from your answer.

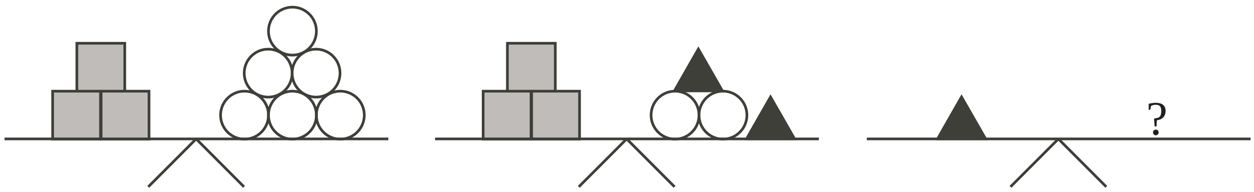
6. Farmer Brown is making a pen for his sheep. For x sheep, he needs an a pen with area $4x$ m^2. If Farmer Brown makes the pen 10 m wide, how long does he have to make the pen to hold:
   a) 5 sheep?
   b) 10 sheep?
   c) 20 sheep?

7. (14 G7 2002) The first six letters of the alphabet are assigned values A = 1, B = 2, C = 3, D = 4, E = 5, and F = 6. The value of a word equals the sum of the values of its letters. For example, the value of BEEF is 2 + 5 + 5 + 6 = 18. Which of the following words has the greatest value?

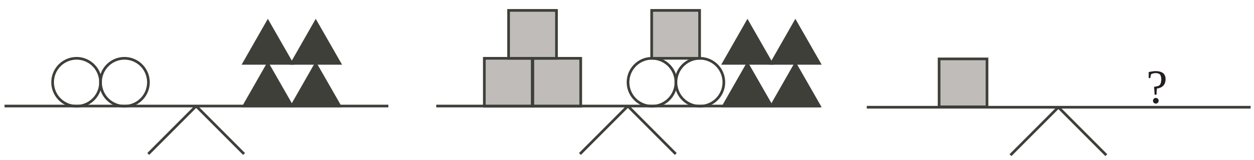
(**A**) BEEF (**B**) FADE (**C**) FEED (**D**) FACE (**E**) DEAF

8. (16 G7 2006) A fraction is equivalent to $\frac{5}{8}$. Its denominator and numerator add up to 91. What is the difference between the denominator and the numerator?

9. a) How many circles will balance the triangle?



b) How many triangles will balance the square?



10. (16 G7 2002) In the following equations, the letters a, b and c represent different numbers.

$$1^3 = 1$$
$$a^3 = 1 + 7$$
$$3^3 = 1 + 7 + b$$
$$4^3 = 1 + 7 + c$$

What is the numerical value of $a + b + c$?

11. (25 G7 2006) Five students wrote a quiz with a maximum score of 50. The scores of four of the students were 42, 43, 46, and 49. The score of the fifth student was N. The average (mean) of the five students' scores was the same as the median of the five students' scores. What is the number of possible values of N?

12. In her latest game, Mary bowled 199 and this raised her average from 177 to 178. To raise her average to 179 with the next game, what must she bowl?

13. (23 G8 2003) In her backyard garden, Gabriella has 12 tomato plants in a row. As she walks along the row, she notices that each plant in the row has one more tomato than the plant before. If she counts 186 tomatoes in total, how many tomatoes are there on the last plant in the row?

14. (25 G8 2009) A list of six positive integers p, q, r, s, t, u satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. The sums of these 15 pairs of numbers are:

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117.$$

Which sum equals $r + s$?

Answers

1. a) 12 b) 38 c)12 d) -5 e) $\frac{2}{3}$ f) 21
2. a) $x = 14$ b) $x = 41$ c) $x = 45$ d) $x = 14$ e) $x = 5$
   f) $x = 3$ g) $x = -12$ h) $x = 0$ i) $x = 7$
3. 18
4. 4
5. The final number is the initial number.
6. a) 2 b) 4 c) 8
7. (**C**)
8. 21
9. a) 2 b) 4
10. 77
11. 3
12. 201
13. 21
14. 54