University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

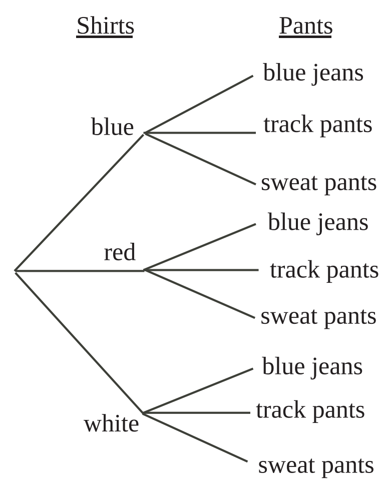
Junior Math Circles
October 7, 2009
Counting

Opening Problem

When Steve wakes up in the morning he has to choose what clothes he is going to wear. He has 3 different shirts (red, blue, and white) and 3 different pants (blue jeans, track pants and sweat pants). How many different outfits can Steve make? Assume each outfit contains at least one of each piece of clothing.

Solution:

Lets list all the possible outfit arrangements.
red and jeans
red and track pants
red and sweats
blue and jeans
blue and track pants
blue and sweats
white and jeans
white and track pants
white and sweats



Steve can make 9 different outfits.
Since the number of shirts and pants was small it was not too difficult to list out all the possible outfits. Sometimes a tree like the one to the right can be helpful when counting.

How would we calculate the number of outfits if Steve included shoes as part of his outfits and had 25 shirts, 12 pants and 5 different shoes? We would not want to list all the possible outfits or draw a tree. In this case we would uses the *Fundamental Counting Principle*.

Fundamental Counting Principle

If one action can be done in a ways and another action can be in b ways and both actions can be done together, then they can be done together in $a \times b$ ways.
The same principle applies when there are more then two actions.

Practice: If Steve has 25 shirts, 12 pants and 5 shoes how many different outfits can me make?

Practice: Chuck Norris is being held captive in a room with 6 windows and 3 doors. How many ways can Chuck Norris escape the room?

Rule of Sum
If one action can be done in m ways, another action can be done in n ways and both actions cannot be done together, then one or the other can be done in $m + n$ ways.

Practice: If you flip a coin three times how many outcomes are possible? An example outcome is heads on the first toss, tails on the second toss, and tails on the third toss.

Problem 2: Factorials, Grouping and the Indirect Method
a.)Friends Ashley, Bob, Dave, and Emily go to the movie theatre to see *The Incredibles*. All four friends sit together in the same row with 4 seats. How many different seating arrangements of the four friends are possible?

Solution:
Any one of the 4 friends could sit in the first seat . Any one of the remaining 3 could sit in the second seat. There are 2 friends who could sit in the third seat. The fourth seat goes to the friend who does not have a seat.

4	3	2	1

Now by applying the fundamental counting principle we have $4 \times 3 \times 2 \times 1 = 24$
Therefore there are 24 possible seating arrangements for the friends at the theatre.

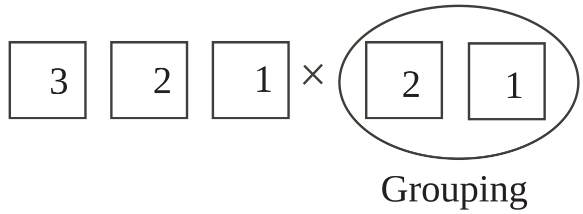
There is a special notation for writing $4 \times 3 \times 2 \times 1$
We define a *factorial* to be $n! = n(n-1)(n-2)(n-3)\ldots(2)(1)$ for a positive integer n.
By definition, $0! = 1$. Then using this notation, $4 \times 3 \times 2 \times 1 = 4!$

Practice: Expand and answer

$3! =$	$6! =$
$5! =$	$1! =$
$2! =$	$0! =$

b.)Ashley and Emily are best friends and plan to be sitting together at the theatre. How many arrangements are there where Ashley and Emily are sitting beside each other?

Solution:
Since Ashley and Emily will be sitting together we can group them together and consider them as one object. In that case we then have 3! arrangements of the pair of girls, Dave and Bob, BUT what about the arrangements of the girls inside their grouping. It could be Ashley then Emily or Emily then Ashley. There are two possible arrangements within the grouping.



$3! \times 2! = 12$
Therefore there are 12 possible arrangements where Ashley and Emily are side by side.

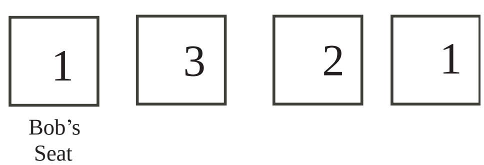
c.)Ashley and Emily had a fight and will not be sitting beside each other. How many arrangements are there where Ashley and Emily are NOT sitting beside each other?

Solution:
We could consider cases when Ashley and Emily are separated by one friend and two friends, but there is an easier way. What is the opposite of Ashley and Emily not sitting beside each other? It is them sitting beside each other. We know how the number of arrangements that have Ashley and Emily sitting beside each other is 12 and that there are a total of 24 different. So if we subtracted the number of opposite arrangements from the total we get our desired answer.
$24 - 12 = 12$
Therefore there are 12 arrangements where Ashley and Emily are not side by side.

The method used for part c.) is called the *indirect method*.
It finds the desired outcomes by taking the TOTAL outcomes - OPPOSITE outcomes.

d.) How many arrangements have Bob sitting in the first (a.k.a the far left) seat?

Solution:
Let's use the squares again to represent the number of friends available for a given seat. First we deal with the restriction, which is Bob has to sit in the far left seat. In seat one only Bob can sit there. Then for the 3 remaining seats it is the same as the in part a.); the number of arrangements of the three friends and three remaining seats.



$$1 \times 3 \times 2 \times 1 = 6$$

Therefore there are 6 arrangements where Bob is seated in the first seat.

Practice: Using all of the letters of the word HOCKEY only once, how many arrangements have the letters K, H and O side-by-side?

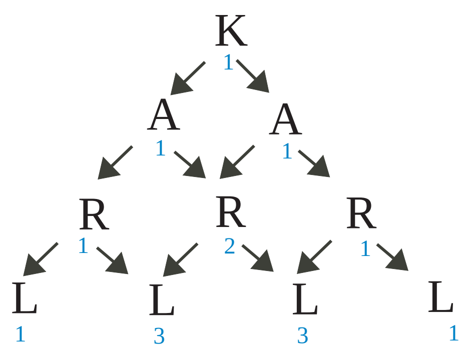
Pascal's Triangle

The triangle below was developed by French mathematician Blaise Pascal. The triangle has each row start and end with 1 and every element is obtained by adding the two entries in the row above to the left and above to the right.

1 ROW 0
1 1 ROW 1
1 2 1 ROW 2
1 3 3 1 ROW 3
1 4 6 4 1 ROW 4
1 5 10 10 5 1 ROW 5
1 6 15 25 15 6 1 ROW 6
⋮

Using Pascal's Triangle to Count Paths

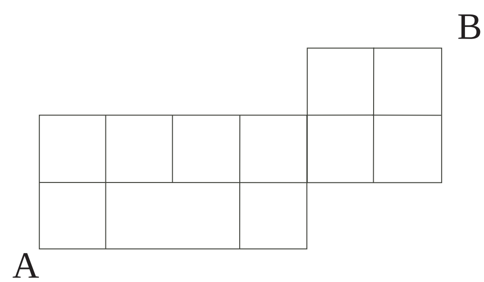
In the diagram, how many paths can be taken to spell "KARL"?



To find the number of paths we move down the KARL tree counting the paths to each letter in the proper order. In the word KARL it starts with K then goes to A. In our tree there is one path from K to each A. Then we move to counting the paths from each A to R, and so on until we reach L. When we reach the bottom of the KARL tree we count up all the paths to each L.

$$1 + 3 + 3 + 1 = 8$$

Practice: How many paths are there from A to B moving only upwards or to the right?



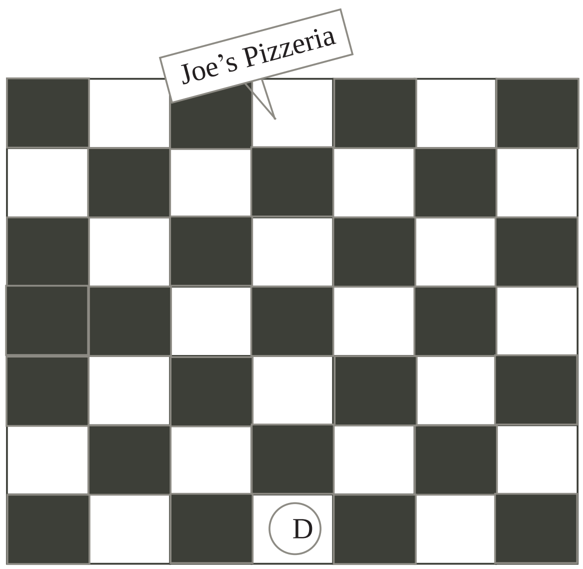
Problem Set

1. Cayli must choose one activity from each of the following groups: art, sports, and music. If there are 2 art choices, 3 sports choices and 4 music choices, how many possible combinations of art, sports, and music choices can Cayli make?

2. a.) Harry, Ron and Neville are having a race on their broomsticks. If there are no ties, in how many different possible orders can they finish?

   b.) Now Hermionie and Ginny decide they are going to join the race. If there are no ties, how many different possible arrangements of the **top three** finishers are possible?

3. Dan is trapped on checkered board where he can only move forward and must stay on white squares. Dan loves pizza and really wants to get a slice from Joe's Pizzeria. How many different routes can Dan take on the checker board to reach the Pizzeria?



4. The whole numbers from 1 to 1000 are written. How many of these numbers have at least two 7's appearing side-by-side?

5. Ashley, Bob, Dave, and Emily liked *The Incredibles* so much that they are going to see it a second time. This time when the friends go to the theatre they all sit in a row of five seats. Assuming they do not have to sit together in the row of five and one of the five seats is left empty, how many different seating arrangements of the four friends are possible in a row with five seats?

6. In how many ways can 6 players be seated on the team bench so that Mike and Kyle are not seated next to each other? Assume Mike and Kyle are included in those 6 players.

7. For a combination lock there are three distinct numbers that must be inputed in a specific order to open the lock. On a lock's dial there are integers from 0 to 59 to choose from. How many different combinations are possible?

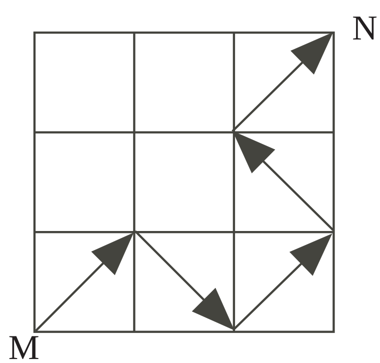
8. While Jim is cleaning out his garage he finds three single-digit house numbers: one 2, one 7 and one 5.

   a.) How many **two-digit** house numbers can Jim create using these three digits?

   b.) How many different house numbers can Jim create using these three digits?

   c.) Jim finds another single digit 5. Including this 5 with the three single digit house numbers he had prior, how many **two-digit** house numbers can he create?

9. Kira can draw a connected path from M to N by drawing arrows along only the diagonals of the nine squares shown. One such possible path is shown. A path cannot pass through the interior of the same square twice. In total, how many different paths can she draw from M to N?



10. How many 5 digit numbers can be made using the integers 1, 2, 3, 4, 5 and 6?

11. How many 6 digit even numbers can be made using the integers 1, 2, 3, 4, 5, 6, 7 and 8?

12. A restricted area at the Acme Corporation requires a 5 digit code, using the integers from 1 to 9, where integers can not be repeated in the code.

    a.)How many codes are possible?

    b.)How many codes contain no even digits?

    c.)How many codes contain at least one even digit?

    d.)How many codes are possible if the code allowed integers to be repeated?

Answers:

Question	Answer
1	24
2a	6
2b	60
3	19
4	19
5	120
6	480
7	205320
8a	6
8b	15
8c	7
9	9
10	720
11	10080
12a	15120
12b	120
12c	15000
12d	59049