University of Waterloo
Faculty of Mathematics

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Mathematics and Computing

# Senior Math Circles
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# Game Theory I

In game theory a *winning position* is a position from which you have a winning strategy, regardless of what your opponent does. A *losing position* is a position where, when you are moving from it, your opponent has a winning strategy.

Rule 1: If from position X we can get to a losing position, then X is a winning position.
Rule 2: If all positions we can get to from position X are winning positions, then X is a losing position.

An important strategy in solving game theory problems is *Working Backwards*. We will start with the classic Game Theory problem.

G1. Two players, David and Wesley begin a game with 33 toothpicks in a pile. On each player's turn, he must take 1, 2, 3, or 4 toothpicks away. Whoever takes the last toothpick and leaves their opponent with no legal move wins. Show that David has a winning strategy assuming he plays first.

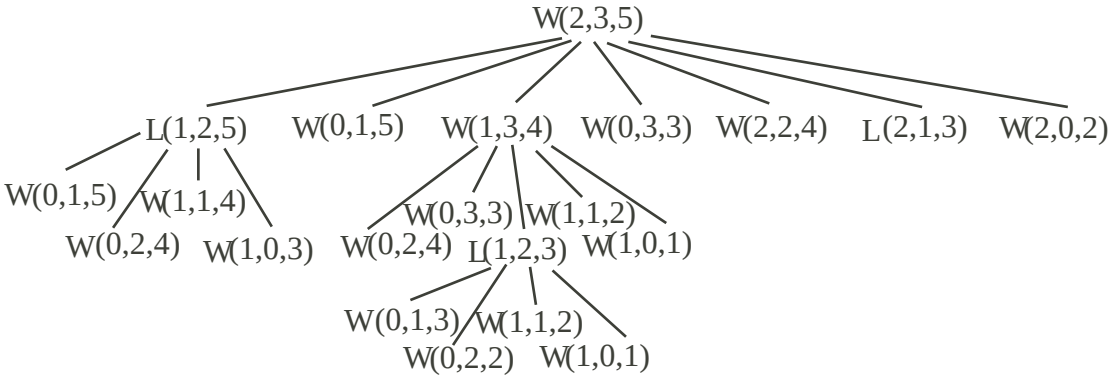
Using the two rules above, we can work backwards from very obvious losing and winning positions to determine whether 33 is a winning or losing position.
0 is an "obvious" losing position, since we would have no possible moves if we receive 0 toothpicks.
1 is a winning position, since we can get to 0, a losing position, from 1. Similarly 2, 3 and 4 are winning positions.
5 is a losing position, since from 5 we can only get to 1, 2, 3 or 4, which are all winning positions. This pattern continues so that every multiple of 5 is a losing position. Therefore, David's winning strategy is to always take away enough toothpicks to leave a multiple of 5 for Wesley.

Now we will try this strategy on something much harder.

G2. Ian and Larry are playing a game with 3 jars of marbles. On each player's turn, they must remove the same number of marbles from each of two different jars. If a player is unable to do so, they lose the game (and their opponent wins).

(a) If it is Ian's turn and the jars contain 2, 3, and 5 marbles respectively, which player has a winning strategy?

The table below shows possible moves. Note that any position with two equal piles or one pile with zero marbles is a winning position. We can work up from the bottom of the table to determine which positions are winning positions and which are losing positions. We can see that Ian has a winning strategy starting by playing either $(1, 2, 5)$ or $(2, 1, 3)$.




(b) If it is Ian's turn and the jars contain 2, 4, and 5 marbles respectively, which player has a winning strategy?

If you construct a chart for this question, you will find that the second player, Larry, has a winning strategy.

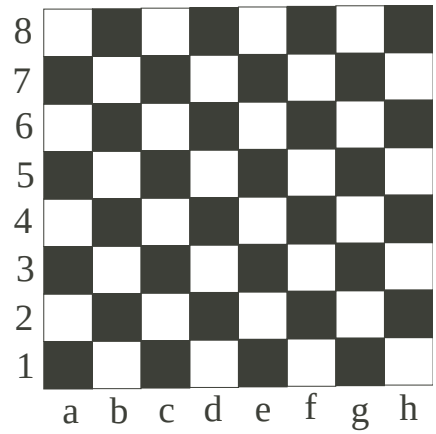
(c) If it is Ian's turn and the jars contain 2, 4, and 6 marbles respectively, which player has a winning strategy?

From $(2, 4, 6)$, Ian can get to $(2, 1, 3)$, which is a losing position as shown in part a). Therefore $(2, 4, 6)$ is a wining position so Ian has a winning strategy.

*Parity* played a part in the last game. During each turn you removed the same number of marbles from two different piles, meaning you were always removing an even number of marbles. As a result the parity (odd or even) of the total number of marbles never changes. From this we can tell that we can use part a) to solve part c) since $2 + 3 + 5 = 10$ and $2 + 4 + 6 = 12$ are both even, but not to solve part b) since $2 + 3 + 6 = 11$ is odd. An *invariant* is something that stays constant throughout the game. Invariants are a useful tool in Game Theory in general.

G3. Consider a rook starting on the lower left corner of a standard $8 \times 8$ chessboard. On any turn the rook may be moved any amount of squares to the right or upward, but not both. Elyot and Jeffrey take turns moving the rook with Elyot going first. The winner is the player who advances the rook to the upper right square, leaving their opponent with no legal moves. What is Jeffrey's winning strategy?

The obvious losing position is h8. The two squares next to it, h7 and g8, are both winning positions because you can move from either to h8. We can then see that g7 is a losing position since you can only move to h7 or g8 from it. If you continue working backwards towards the bottom left corner of the board you will find that all of the positions along the diagonal from a1 to h8 are losing positions, and all others are winning positions. Therefore, Jeffrey's winning strategy is to always move the rook to a square on the diagonal.




G4. *(Nim Variant 1)* Malcolm and Steven are playing a game with piles of beans. On each player's turn, they must remove at least 1, and at most 6 beans from one of the piles. It is Malcolm's turn and there are two piles of beans; one has 25 while the other has 27. The winner is the player to take the last bean, leaving their opponent with no legal move.

(a) Who has the winning strategy?

The losing positions for this game are positions where there are the same number of beans in each pile. As a result, Malcolm has the winning strategy. On his first turn he will take 2 beans from the larger pile leaving two piles of 25. On the remaining turns he will take the same number of beans as Steven, but from the opposite pile.

(b) What if the piles had 25 and 47 beans instead?

If Malcolm first takes one bean from the pile of 47, then uses his strategy from a) and takes the same number of beans as Steven from the opposite pile, the pile that starts with 25 will eventually run out of beans. At this point, there are 21 beans left in a single pile and the game is very similar to G1, except that players can take up to 6 beans each turn. Therefore the losing positions are multiples of 7 and so Malcolm's winning strategy is to always leave a multiple of 7 beans for Steven.

G5. *(Nim Variant 2)* Stephen and Mike are playing a game with 4 stacks of cards. On each player's turn, they must remove some number of cards from any one stack. The player who takes the last card and leaves their opponent with no legal move wins! It is Stephen's turn and the stacks have 11, 12, 14, and 15 cards, respectively. Show that Stephen has a winning strategy.

See next week's lecture for more detail. (Steven can win this game by forcing the Nim-Sum back to 0 on each of his turns.)

The last concept we will talk about today is *Isomorphic Games*. Isomorphic Games are games whose rules are fundamentally the same, although the objects used to play them may be very different. Since isomorphic games have the same rules, they also have the same winning strategy.
Think about the following game; does it look familiar?

G6. *(Ninety-nine)* Karina and Christina alternately write numbers on the blackboard. On each move, a player can increase the tens digit or the ones digit but not both. The starting number is 11 and the player who writes 99 and leaves their opponent with no legal move wins. It is Karina's turn. Which player has a winning strategy?

This game is isomorphic to G3. Increasing the tens digit is the same as moving the rook upwards. Increasing the ones digit is the same as moving the rook right. The diagonal on the chess board is the same as the multiples of 11 in this game. Since these two games are isomorphic, the second player, Christina has a winning strategy which is to always increase the ones or tens digit so that the number is a multiple of 11. Even though this game is played with numbers, and G3 is played on a chess board, their rules and winning strategies are the same.

## Problem Set

1. Consider $G6$ above.
   (a) What if we play *Nine hundred ninety-nine* with analogous rules?
   (b) What if we play *Nine thousand nine hundred ninety-nine*?
2. *(Hypatia '03)* Xavier and Yolanda are playing a game starting with some coins arranged in piles. Xavier always goes first, and the two players take turns removing one or more coins from any one pile. The player who takes the last coin wins.
   (a) If there are two piles of coins with 3 coins in each pile, show that Yolanda can guarantee that she always wins the game.
   (b) If the game starts with piles of 1, 2, and 3 coins, explain how Yolanda can guarantee that she always wins the game.
3. *(Hypatia '05)* Gwen and Chris are playing a game. They begin with a pile of toothpicks, and use the following rules:
   - The two players alternate turns
   - On any turn, the player can remove 1, 2, 3, 4, or 5 toothpicks from the pile
   - The same number of toothpicks cannot be removed on two different turns (once $x$ toothpicks have been removed in a player's turn, no player may choose to remove $x$ toothpicks on their turn for the rest of the game)
   - The last person who is able to play wins, regardless of whether there are any toothpicks remaining in the pile

   (a) Suppose the game begins with 11 toothpicks. Gwen begins by removing 3 toothpicks. Chris follows and removes 1. Then Gwen removes 4 toothpicks. Explain how Chris can win the game.
   (b) Suppose the game begins with 10 toothpicks. Gwen begins by removing 5 toothpicks. Explain why Gwen can always win, regardless of what Chris removes on his turn.
   (c) Suppose the game begins with 9 toothpicks. Gwen begins by removing 2 toothpicks. Explain how Gwen can always win, regardless of how Chris plays.

4. There is a strip of paper with 10 squares in a horizontal line. The two leftmost squares each contain a coin. On each player's turn, they can move either coin any number of spaces to the right as long as it doesn't jump over or land on the other coin. The last player to be able to move wins.

   (a) Does the first player or the second player have a winning strategy?

   (b) What if the three leftmost squares each contain a coin?

   (c) What if squares 1, 3, 5 (from the left) each contain a coin?

5. *(Hypatia '04)*

   (a) 1 green, 1 yellow and 2 red balls are placed in a bag. Two balls of different colours are selected at random. These two balls are then removed and replaced with one ball of the third colour. Enough extra balls of each colour are kept to the side for this purpose. This process continues until there is only one ball left in the bag, or all of the balls are the same colour. What is the colour of the ball or balls that remain at the end?

   (b) 3 green, 4 yellow and 5 red balls are placed in a bag. If a procedure identical to that in part (a) is carried out, what is the colour of the ball or balls that remain at the end?

   (c) 3 green, 4 yellow and 5 red balls are placed in a bag. This time, two balls of different colours are selected at random, removed, and replaced wtih two balls of the third colour. Show that it is impossible for all of the remaining balls to be the same colour, no matter how many times this process is repeated.

6. *(Hypatia '07)* Olayuk has four pails labelled $P$, $Q$, $R$, and $S$, each containing some marbles. A legal move is to take one marble from each of three of the pails and put the marbles into the fourth pail.

   (a) Initially, the pails contain 9, 9, 1, and 5 marbles. Describe a sequence of legal moves that results in 6 marbles in each pail.

   (b) Suppose that the pails initially contain 31, 27, 27, and 7 marbles. After a number of legal moves, each pail contains the same number of marbles.

      i. Describe a sequence of legal moves to obtain the same number of marbles in each pail.

      ii. Explain why at least 8 legal moves are needed to obtain the same number of marbles in each pail.

   (c) Beginning again, the pails contain 10, 8, 11, and 7 marbles. Explain why there is no sequence of legal moves that results in an equal number of marbles in each pail.