Sequences and Series

Sequences Week 2

Consider the sequence

$$3, 2, 1, 0, 3, 2, 1, 0, 3, 2, 1, 0, 3, 2, 1, 0, \dots$$

What is the 101^{st} number in the sequence?

This is an example of a cyclic sequence, or a sequence which has a cycle in it which continually repeats. Since there are four numbers in the cycle, then the fourth term, and every fourth term after that, will be a 0. Thus the 100^{th} term will be a 0, since 100 is a multiple of 4. Thus the 101^{st} term is a 3.

Series

A *series* is simply the sum of a sequence.

Given the sequence $\{a_n\}$, the series associated with it would be:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots$$

Arithmetic Series

An *arithmetic series* is the sum of an *arithmetic sequence*.
For example: the arithmetic sequence $\{2 + 3(n - 1)\} = 2, 5, 8, 11, \dots$ gives us the arithmetic series

$$2 + 5 + 8 + 11 + \dots$$

This arithmetic series has *initial term* 2, and *common difference* 3.

In general we will represent the *initial term* using a, and the *common difference* using d. Thus a general arithmetic series is:

$$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + \cdots + (a + (n - 2)d) + (a + (n - 1)d) + \dots$$

Let S_n represent the sum of the first n terms of the series. It is often useful to express S_n in terms of n. For arithmetic series this is always possible.

We calculate the sum of the terms from a_1 to a_n as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 2)d) + (a + (n - 1)d)$$
$$\text{and } S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + 2d) + (a + d) + a$$

If we add these together term by term we obtain:

$$2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + \cdots + (2a + (n - 1)d) + (2a + (n - 1)d) = n(2a + (n - 1)d)$$

and so we have that

$$S_n = \frac{n(2a + (n - 1)d)}{2}$$

Example: using the series from above:

$$2 + 5 + 8 + 11 + \dots$$

we have $a = 2$, and $d = 3$, and if we want the sum of the first 100 term, ie $n = 100$, we get

$$S_{100} = \frac{100(2(2) + (100-1)3)}{2} = \frac{100(4 + (99)3)}{2} = \frac{100(4 + 297)}{2} = 50(301) = 15050.$$

Geometric Series

An *geometric series* is the sum of an *geometric sequence*.
For example: the geometric sequence $\{3 * 2^{(n-1)}\} = 3, 6, 12, 24, \dots$ gives us the geometric series

$$3 + 6 + 12 + 24 + \dots$$

This geometric series has *initial term* 3, and *common ratio* 2.

In general we will represent the *initial term* using a, and the *common ratio* using r. Thus a general geometric series is:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{(n-2)} + ar^{(n-1)} + \dots$$

Let S_n represent the sum of the first n terms of the series. It is often useful to express S_n in terms of n. For geometric series this is always possible.

We calculate the sum of the terms from a_1 to a_n as follows:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{(n-2)} + ar^{(n-1)}$$
$$\text{and } r \times S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{(n-1)} + ar^n$$

If we subtract these we obtain:

$$\begin{aligned}(1-r) \times S_n &= a + 0 * ar + 0 * ar^2 + 0 * ar^3 + 0 * ar^4 + \cdots + 0 * ar^{(n-1)} - ar^n \\ &= a - ar^n\end{aligned}$$

and so we have that

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

Example: using the series from above:

$$3 + 6 + 12 + 24 + \dots$$

we have $a = 3$, and $r = 2$, and if we want the sum of the first 10 term, ie $n = 10$, we get

$$S_{10} = \frac{3(1-2^{10})}{(1-2)} = \frac{3(1-1024)}{(1-2)} = \frac{3(-1023)}{-1} = 3069.$$

Some useful results. For any series:

$$S_n = S_{(n-1)} + a_n, \text{ so } S_n - S_{(n-1)} = a_n$$
$$S_1 = a_n$$

Something a little bit beyond where we are, and thus not really needed for our discussion.

Consider S_n for a geometric series:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{(1-r)} \\ &= \frac{a - ar^n}{(1-r)} \\ &= \frac{a}{(1-r)} - \frac{r^n}{(1-r)} \\ &= \frac{a}{(1-r)} - \frac{ar^n}{(1-r)} \\ &= \frac{a}{(1-r)} - \frac{a}{(1-r)}r^n \end{aligned}$$

Now if $|r| < 1$, that is if $-1 < r < 1$, then $|r^n|$ gets smaller as n gets bigger. Thus as we take more and more terms into the sum of the series the value of r^n, and thus $\frac{a}{(1-r)}r^n$, gets very very small. In fact, it approaches zero, and thus when we add infinitely many terms of the series together, the sum becomes $\frac{a}{(1-r)}$, and we say $S_\infty = \frac{a}{(1-r)}$.