Sequences and Series

Sequences

Initial definition: a list of numbers.

Asking a group of people to each contribute a number resulted in a sequence like:

$$3, 5, 1, 27, 13, 29, 85, 9, \ldots$$

Such a sequence can best be described as a *random* sequence. Although random sequences have a place in mathematics, they are not what we normally think about when we refer to a sequence. We also usually, and will here, restrict ourselves to sequences of *Real* numbers, avoiding *Complex* numbers.

When we think of a sequence of numbers, we think of it as having some order, or predictability, or rule, so that we can determine new terms (numbers) in the sequence.

Thus, a modified definition could be:
A *sequence of real numbers* is a list of real numbers which follow some rule for being part of the sequence.

How do we define a sequence?

We could give the first few terms and see if that is enough. For example...

2, 4, ...

The following were suggested as next terms.
6, the sequence is simply the even positive integers,

8, the sequence is defined by doubling a term to get the next term,

16, the sequence is defined by squaring a term to get the next term.

Thus we see that each of the above third numbers in the sequence make sense for a specific rule which defines the sequence.

Consider the sequence which begins

$$2, 4, 8, 16, \ldots$$

What is the next term or number in the sequence? Many people would say 32, some might say 64. If you were told that the next number(term) in the sequence is 3, can you determine the rule?

The rule is defined by sets of 4 numbers as follows:

$$2^1, 2^2, 2^3, 2^4, 3^1, 3^2, 3^3, 3^4, 4^1, 4^2, 4^3, 4^4, 5^1, \ldots$$

The first four positive powers of the natural numbers taken in order.

It is fairly clear that care must be taken to carefully define a sequence so that there can be no mistake about the terms of the sequence.

Before we look at ways to define sequences, we need some notation:

We can denote terms in a sequence by subscripted letters, for example: a_n would be the n^{th} term of a sequence.
So the first term would be a_1, the second term a_2, the hundredth term a_{100}, and so on.

Now we can define a sequence by giving a rule(function) for the n^{th} term, and, unless we are told otherwise, the values of n are the natural numbers in order from 1.
A short notation for the sequence with general term a_n is $\{a_n\}$.

An example is the following:

$$\{a_n\} = \{2 + (n-1)2\} \text{ give the sequence:}$$

$$a_1 = 2 + (1-1)2 = 2$$
$$a_2 = 2 + (2-1)2 = 4$$
$$a_3 = 2 + (3-1)2 = 6$$

and so on forever.

The same sequence could be defined recursively as follows:

$$a_{n+1} = a_n + 2, a_1 = 2$$

Here we explain how to get from one term to the next, and then give the first term.
Both of the above definitions give the sequence:

$$\{a_n\} = 2, 4, 6, 8, 10, 12, \ldots$$

This is an example of an *arithmetic* series, as we get from one term to the next term by adding the same constant, in this case 2, to the current term. The constant which is added to each term to obtain the next term is called the *common difference* since it is the difference between any two consecutive terms.

Another example is:

$$\{a_n\} = \{2^n\} \text{ give the sequence:}$$

$$a_1 = 2^1 = 2$$
$$a_2 = 2^2 = 4$$
$$a_3 = 2^3 = 8$$

and so on forever.

The same sequence could be defined recursively as follows:

$$a_{n+1} = 2a_n, a_1 = 2$$

Both of the above definitions give the sequence:

$$\{a_n\} = 2, 4, 8, 16, 32, \ldots$$

This is an example of an *geometric* series, as we get from one term to the next term by multiplying the current term by the same constant, in this case 2. The constant which each term is multiplied by to obtain the next term is called the *common ratio* since it is the ratio of any two consecutive terms.

If we are given a recursive definition for the sequence and we wish to find the formula in terms of n for a_n, there are a number of techniques which may have to be used. For simple recursions such as:

$$a_{n+1} = a_n + 3(n), a_1 = 1$$

we can employ a summing approach as follows.

$$\begin{aligned} a_n &= a_{n-1} + 3(n-1) \\ a_{n-1} &= a_{n-2} + 3(n-2) \\ a_{n-2} &= a_{n-3} + 3(n-3) \\ a_{n-3} &= a_{n-4} + 3(n-4) \\ &\vdots \\ a_3 &= a_2 + 3(2) \\ a_2 &= a_1 + 3(1) \end{aligned}$$

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summing gives $a_n + a_{n-1} + \cdots + a_3 + a_2 = a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + 3(1 + 2 + \cdots + (n-1))$

This simplifies to

$$a_n = a_1 + 3(1 + 2 + \cdots + (n-1))$$

or

$$a_n = 1 + 3(1 + 2 + \cdots + (n-1)).$$

We need to sum the positive integers from 1 to $(n-1)$.

$$\begin{aligned} \text{Let } S &= 1 + 2 + \cdots + (n-2) + (n-1) \\ \text{so } S &= (n-1) + (n-2) + \cdots + 2 + 1 \end{aligned}$$

If we add these together we obtain:

$$2S = n + n + \cdots + n + n = n(n-1)$$

and so we have that

$$S = \frac{n(n-1)}{2}$$

Thus we obtain

$$a_n = 1 + 3\frac{n(n-1)}{2} = \frac{3}{2}n^2 - \frac{3}{2}n + 1$$

for all $n \geq 1$.