Number Theory # 2

Greatest Common Divisor and Lowest Common Multiple

Recall that the greatest common divisor of two natural numbers a and b is the largest positive number that is a factor of both a and b. The lowest common multiple of a and b is the smallest positive number that is a multiple of a and b. We denote the greatest common divisor of a and b by

$$\gcd(a, b)$$

and the lowest common multiple of a and b by

$$\operatorname{lcm}(a, b)$$

REMARK: Some mathematicians denote the greatest common divisor of a and b just as (a, b).

EXAMPLE Find the gcd and lcm of the following pairs.

(a) $a = 40$ and $b = 28$

Solution: We can list all of the factors of each

$40 : 1, 2, 4, 5, 8, 10, 20, 40$

$28 : 1, 2, 4, 7, 14, 28$

Thus, $\gcd(40, 28) = 4$.

For multiples we have

$40 : 40, 80, 120, 160, 200, 240, 280, ...$

$28 : 28, 56, 84, 112, 140, 168, 196, 224, 252, 280, ...$

So, $\operatorname{lcm}(40, 28) = 280$.

(b) $a = 81$ and $b = 256$

Solution: The factors of a are all multiples of 3 and the factors of b are all even, so $\gcd(81, 256) = 1$. We find that $\operatorname{lcm}(81, 256) = 81 \times 256 = 20736$.

What if we wanted to find the gcd and lcm of $a = 12!$ and $b = 100^3$. This would be a little more difficult. Listing all the factors would be very bad... I don't want to even think about writing out the multiples! We need to find another way of doing this. But how? With PRIMES! Prime numbers are the building blocks of the numbers, so they should be able to help us. Let us rewrite the first example above in terms of prime numbers (i.e. prime factorization).

The factors of $a = 40 = 2^3 \times 5$ and $b = 28 = 2^2 \times 7$ are

$2^3 \times 5 : 1, 2, 2^2, 5, 2^3, 2 \times 5, 2^2 \times 5, 2^3 \times 5$

$2^2 \times 7 : 1, 2, 2^2, 7, 2 \times 7, 2^2 \times 7$

So, the $\gcd(a, b) = 2^2$. That is clear from the prime factorization. The largest common factor they have is 2^2.

What about the lcm? The multiples are:

$2^3 \times 5 : 2^3 \times 5, 2^4 \times 5, 2^3 \times 3 \times 5, 2^5 \times 5, 2^3 \times 5^2, 2^4 \times 3 \times 5, 2^3 \times 5 \times 7, \ldots$

$2^2 \times 7 : 2^2 \times 7, 2^3 \times 7, 2^2 \times 3 \times 7, 2^4 \times 7, 2^2 \times 5 \times 7, 2^3 \times 3 \times 7, 2^2 \times 7^2, 2^5 \times 7, 2^3 \times 3^2 \times 7, 2^3 \times 5 \times 7, \ldots$

So, $\operatorname{lcm}(a, b) = 2^3 \times 5 \times 7$. The lowest common multiple is the smallest number that has all the prime factors from both numbers.

Thus, the gcd of two numbers is the largest part of their prime factorization that they have in common.

The lcm of two numbers is the number which has all the prime factors from both numbers and no other factors.

Mathematically, we write:

If $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, then

$$\gcd(a, b) = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$$

where $e_i = \min(\alpha_i, \beta_i)$, and

$$\operatorname{lcm}(a, b) = p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$$

where $f_i = \min(\alpha_i, \beta_i)$.

EXAMPLE Let $a = 784$ and $b = 1400$. Find $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$.

Solution: Writing out the prime factorizations we get

$$a = 2^4 \times 7^2 \qquad 1400 = 2^3 \times 5^2 \times 7$$

Thus, $\gcd(784, 1400) = 2^3 \times 7 = 56$ and $\operatorname{lcm}(784, 1400) = 2^4 \times 5^2 \times 7^2 = 19600$.

EXERCISE 1: Find the gcd and lcm of the following pairs.

1. $a = 756$ and $b = 726$

2. $c = 478$ and $d = 675$
3. $e = 12!$ and $f = 100^3$

Solution: $a = 2^2 \times 3^3 \times 7$ and $b = 2 \times 3 \times 11^2$ so

$$\gcd(a, b) = 2 \times 3 = 6, \quad \operatorname{lcm}(a, b) = 2^2 \times 3^3 \times 7 \times 11^2 = 91476$$

$c = 2 \times 239$ and $d = 3^3 \times 5^2$, so

$$\gcd(c, d) = 1, \quad \operatorname{lcm}(c, d) = 2 \times 239 \times 3^3 \times 5^2 = 322650$$

$e = 2^{10} \times 3^5 \times 5^2 \times 7 \times 11$ and $f = 2^6 \times 5^6$ so

$$\gcd(e, f) = 2^6 \times 5^2 = 1600, \quad \operatorname{lcm}(e, f) = 2^{10} \times 3^5 \times 5^6 \times 7 \times 11$$

A large part of mathematics is looking for patterns. Since both the gcd and the lcm are calculated by using the powers in the exponents of the prime factorization, there must be a relationship between them. From the examples and exercises above, can you see the relationship?

We have that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$. This gives us another way of calculating the lcm assuming that we know the gcd. In particular,

$$\operatorname{lcm}(a, b) = \frac{ab}{\gcd(a, b)}$$

How does this help us? Consider the following problem:

Neptune does one complete revolution of the sun every 1444764 hours the earth does one complete rotation every 8766 hours. How often does the sun, Earth, and Neptune line up?

To answer this problem, we really need to calculate $\operatorname{lcm}(1444764, 8766)$. Finding the prime factorization of 1444764 is likely very hard (try it, you won't like it). Instead, if we will look at a much better way of finding the gcd of two numbers. Then, we will be able to easily find the lcm using our formula above.

The Euclidean Algorithm

We will make finding the gcd of two numbers easier by proving a reduction formula. That is, a formula which allows us to use smaller numbers to calculate the same thing.

THEOREM 1: Let a and b be positive integers. Then, $\gcd(a, b) = \gcd(a, b - qa)$ for any $q \in \mathbb{Z}$.

Proof: Let $d = \gcd(a, b)$. To prove that d is the gcd of a and $b - qa$ we need to show that d is a factor of a and $b - qa$ and that every other factor of a and $b - qa$ is smaller than d.

Since $d = \gcd(a, b)$ we have that d is a factor of a and a factor of b, so there exists numbers s and t such that $a = sd$ and $b = td$. Then,

$$b - qa = td - qsd = d(t - qs)$$

so d is also a factor of $b - qa$.

Let e be any number that is a factor of a and $b - qa$. Then, there exists numbers m and n such that $a = me$ and $b - qa = ne$. Then,

$$b = ne + qa = ne + qme = e(n + qm)$$

So, e is a factor of a and b. Thus, e cannot be larger than d as otherwise, it would be the greatest common divisor of a and b.

Thus, $\gcd(a, b) = \gcd(a, b - qa)$. □

Thus, to find the gcd of a and b, instead of using a and b we can subtract multiples of the smaller from the larger so that we have smaller numbers. Repeatedly doing this is called the Euclidean Algorithm. This is best demonstrated by example.

EXAMPLE Find $\gcd(24, 210)$.

Solution: We have that $210 = 8 \times 24 + 18$. Thus, taking $q = 8$, the theorem gives

$$\gcd(24, 210) = \gcd(24, 210 - 8 \times 24) = \gcd(24, 18)$$

We can then repeat the procedure. Since $24 = 1 \times 18 + 6$ we get

$$\gcd(24, 210) = \gcd(24, 18) = \gcd(18, 24 - 18) = \gcd(18, 6) = 6$$

This method is called the Euclidean Algorithm. Often we write it without applying the theorem at each step. We will demonstrate this with a couple of examples.

EXAMPLE Calculate $\gcd(945, 399)$.

Solution: We have

$$\begin{aligned}
945 &= 2 \times 399 + 147 \\
399 &= 2 \times 147 + 105 \\
147 &= 1 \times 105 + 42 \\
105 &= 2 \times 42 + 21 \\
42 &= 2 \times 21 + 0
\end{aligned}$$

Therefore,

$$\gcd(945, 399) = \gcd(399, 147) = \gcd(147, 105) = \gcd(105, 42) = \gcd(42, 21) = 21$$

EXERCISE 2: Calculate the following

1. $\gcd(54, 315)$
2. $\gcd(70983, 35491)$

EXAMPLE Find $\operatorname{lcm}(1444764, 8766)$.

Solution: We first find $\gcd(1444764, 8766)$.

$$\begin{aligned}
1444764 &= 164 \times 8766 + 7140 \\
8766 &= 7140 + 1626 \\
7140 &= 4 \times 1626 + 636 \\
1626 &= 2 \times 636 + 354 \\
636 &= 354 + 282 \\
354 &= 282 + 72 \\
282 &= 3 \times 72 + 66 \\
72 &= 66 + 6 \\
66 &= 11 \times 6 + 0
\end{aligned}$$

Thus, $\gcd(1444764, 8766) = 6$ and hence

$$\operatorname{lcm}(1444764, 8766) = \frac{1444764 \times 8766}{6}$$

Problems

1. Find the gcd and lcm of a and b where

   (a) $a = 1296$, $b = 1000$

   (b) $a = 10!$, $b = 6^8$

   (c) $a = 241$, $b = 197$

   (d) $a = 11850$, $b = 600$

2. The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Prove that the gcd of two consecutive Fibonacci numbers is 1.

3. Mercury takes 2111 hours to complete one revolution of the sun, while Venus takes 5393 hours.

   (a) How often does the sun, Mercury, and Venus line up?

   (b) How often does the sun, Mercury, Venus, the Earth, and Neptune line up?