

# Intermediate Math Circles - Analytic Geometry I

John Galbraith  
(j5galbra@uwaterloo.ca)

Centre for Education in Mathematics and Computing  
Faculty of Mathematics  
University of Waterloo  
Waterloo, Canada

March 23, 2011

# 1. Cartesian Plane

We use a coordinate system to allow us to translate a geometric problem into an algebraic problem.

We bring a lot to the table: angle properties and theorems, similar and congruent triangles, etc. In the fall you examined Euclidean Geometry learning 10 facts about angles, 7 facts about side lengths, and 6 facts about circles.

Credit for developments in the area of Analytic Geometry go to Rene Descartes. Descartes is also credited with the phrase: "*I think, therefore I am.*"

# 1. Cartesian Plane

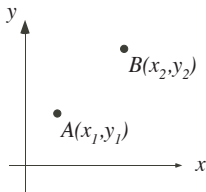


# 1. Cartesian Plane

## (a) Coordinate System and Points

The coordinate system requires an origin, an  $x$ -axis, and a  $y$ -axis with which you should be familiar. Later in math we will extend to a third dimension.

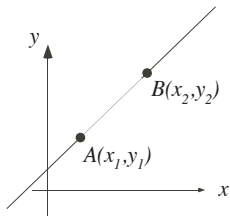
A point is a specific location on the Cartesian Plane. The diagram shows two points  $A$  and  $B$ .



# 1. Cartesian Plane

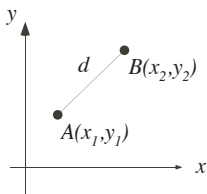
## (b) Lines

A straight line can be drawn through two points. It has no beginning and no end.



# 1. Cartesian Plane

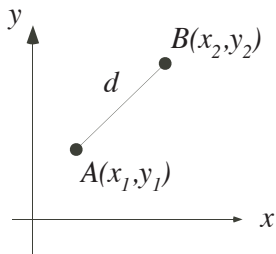
## (c) Line Segments



$AB$  is a line segment with endpoints  $A$  and  $B$ . A line segment is different from a line because it has a fixed length.

We are often interested in finding distances.

## 2. Distance Between Two Points

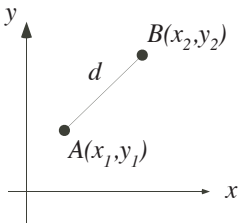


If  $d$  is the distance between two points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Can we prove this result? *Proof is shown in the video online.*

## 2. Distance Between Two Points



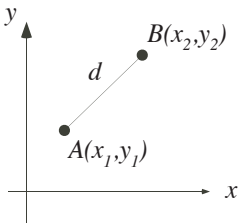
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Does our formula allow for horizontal and vertical distances?

*This result is shown in the video online.*



## 2. Distance Between Two Points



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Does the order of the points matter when we use the formula? *Result is shown in the video online.*

## 2. Distance Between Two Points

### Problem (i)

The line segment joining  $A(2, 6)$  to  $B(8, -2)$  forms the base of isosceles  $\triangle ABP$ . The  $x$ -coordinate of the third vertex  $P$  is  $-5$  and  $AP = BP$ . Determine the  $y$ -coordinate of point  $P$ .

*Solution is shown in the video online.*

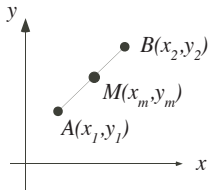
## 2. Distance Between Two Points

### Problem (ii)

Determine the centre of a circle which passes through points  $P(0, 3)$ ,  $Q(2, -1)$ , and  $R(9, 0)$ .

*Solution is shown in the video online.*

### 3. Midpoint Between Two Points



If  $M(x_m, y_m)$  is the midpoint of  $AB$ , then

$$M = \text{midpoint}(AB) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Can we prove this? *Proof is shown in the video online.*

### 3. Midpoint Between Two Points

#### Problem (iii)

Points  $P$ ,  $Q$ , and  $R$  divide the line segment from  $A(0, 2)$  to  $C(6, 0)$ , in that order, into four equal parts. Determine the coordinates of the three points,  $P$ ,  $Q$ , and  $R$ . *Solution is shown in the video online.*

## 4. Slope of a Line / Line Segment

### (a) Definition and Formula

Slope is a measure of the steepness of a line (or line segment).

Slope is generally represented by the letter  $m$ .

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

If you have completed the grade 9 mathematics course you should be very familiar with the concept of slope!

## 4. Slope of a Line / Line Segment

### (b) Special Cases

#### (i) Slope of Horizontal Lines

For horizontal lines, the  $y$ -coordinate is constant meaning  $y_2 = y_1$  or  $y_2 - y_1 = 0$ .

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$

## 4. Slope of a Line / Line Segment

### (b) Special Cases

(i) Slope of Horizontal Lines:  $m = 0$

(ii) Slope of Vertical Lines

For vertical lines, the  $x$ -coordinate is constant meaning  $x_2 = x_1$  or  $x_2 - x_1 = 0$ .

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0}$$

Since division by zero is undefined, the slope of a vertical line is undefined.



## 4. Slope of a Line / Line Segment

### (b) Special Cases

- (i) Slope of Horizontal Lines:  $m = 0$
- (ii) Slope of Vertical Lines:  $m$  is undefined
- (iii) Slope of Parallel Lines

Parallel Lines have the same slope. For most students this is intuitively obvious. We will not spend much time here.

If  $l_1 \parallel l_2$  then  $m_1 = m_2$ .

## 4. Slope of a Line / Line Segment

### (b) Special Cases

- (i) Slope of Horizontal Lines:  $m = 0$
- (ii) Slope of Vertical Lines:  $m$  is undefined
- (iii) Slope of Parallel Lines:  $m_1 = m_2$
- (iv) Slope of Perpendicular Lines

Perpendicular Lines have negative reciprocal slopes.

$$\text{If } l_1 \perp l_2 \text{ then } m_1 = \frac{-1}{m_2} \text{ or } m_1 \times m_2 = -1.$$

## 4. Slope of a Line / Line Segment

### (b) Special Cases Summary

- (i) Slope of Horizontal Lines:  $m = 0$
- (ii) Slope of Vertical Lines:  $m$  is undefined
- (iii) Slope of Parallel Lines:  $m_1 = m_2$
- (iv) Slope of Perpendicular Lines:  $m_1 = \frac{-1}{m_2}$  or  $m_1 \times m_2 = -1$

## 4. Slope of a Line / Line Segment

### Problem (iv)

$\triangle BAH$  has vertices at  $B(-1, 9)$ ,  $A(a, 1)$  and  $H(-7, -4)$  such that  $\angle BAH = 90^\circ$ . Determine the value of  $a$ .

Present two different solutions. *Solutions are shown in the video online.*

## 5. Exercises: Online Solutions available by weekend

### Problem (1)

Three points are *collinear* if they all lie on a straight line. Show that  $P(-12, 1)$ ,  $Q(-4, -3)$  and  $R(6, -8)$  are collinear.

- (a) Use a slope argument to show collinearity.
- (b) Use a distance argument to show collinearity.

### Problem (2)

The point  $A(-2, y)$  is on a line that passes through the points  $T(0, -2)$  and  $W(4, 0)$ . Determine the value of  $y$ .

### Problem (3)

$\triangle ABC$  has vertex  $A$  on the  $x$ -axis at  $-2$  and vertex  $C$  on the  $x$ -axis at  $8$ . The third vertex  $B$  is on the  $y$ -axis at  $b$  such that  $\angle ABC = 90^\circ$ . Determine all possible values of  $b$ .

## 5. Exercises: Online Solutions available by weekend

### Problem (4)

A point  $W$  is located on the  $x$ -axis so that it is 13 units from the point  $R(7, 5)$ . Find the coordinates of point  $W$ .

### Problem (5)

The points  $A$  and  $B$  are located in the first quadrant, equidistant from the origin,  $O$ . If the slope of  $OA$  is 7 and the slope of  $OB$  is 1, determine the slope of  $AB$ .

## 5. Exercises: Online Solutions available by weekend

### Problem (6)

The vertices of  $\triangle ABC$  are  $A(-2, -11)$ ,  $B(10, 5)$  and  $C(12, 3)$ .

- Determine the midpoint  $M$  of line segment  $AB$ .
- Show that  $AM = MB = MC$ . This will prove that  $M$  is the centre of a circle containing points  $A$ ,  $B$  and  $C$  on the circumference.
- Show that  $\angle ACB = 90^\circ$ .

### Problem (7)

The line segment  $AB$ , where  $A$  is  $(2, -4)$  and  $B$  is  $(10, 8)$ , is divided at  $Q$  in the ratio  $3 : 5$ . Find the coordinates of  $Q$ .