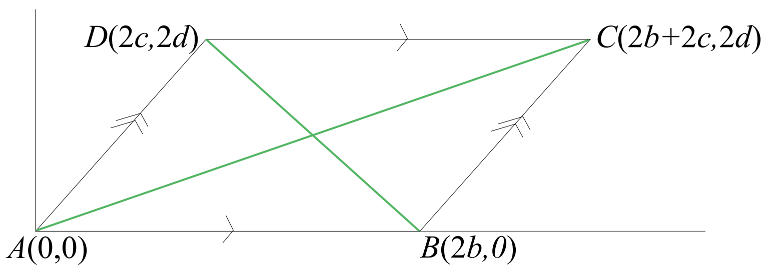
Solutions

Problem 1

Find the midpoint of line AC and line BD:

$$M_{AC} = \left(\frac{2b+2c}{2}, \frac{2d}{2}\right) = (b+c, d) \quad \text{and} \quad M_{BD} = \left(\frac{2b+2c}{2}, \frac{2d}{2}\right) = (b+c, d)$$

Therefore, since the midpoints of each line are the same point, it is the point of intersection, and so the diagonals of a parallelogram bisect each other.

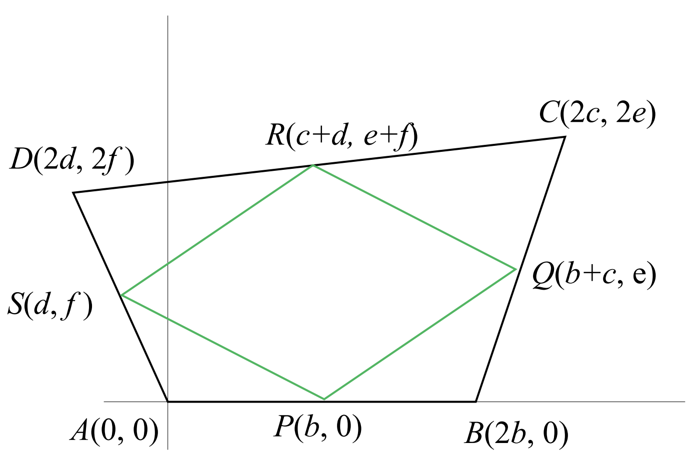


Problem 2

$$m_{PQ} = \frac{e}{(b+c)-b} = \frac{e}{c} \quad \text{and} \quad m_{SR} = \frac{(e+f)-f}{(c+d)-d} = \frac{e}{c}$$

$$m_{QR} = \frac{(e+f)-e}{(c+d)-(b+c)} = \frac{f}{d-b} \quad \text{and} \quad m_{PS} = \frac{f}{d-b}$$

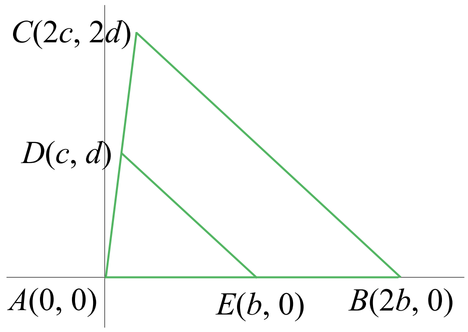
$\therefore m_{PQ} = m_{SR}$ and $m_{QR} = m_{PS}$, so $PQRS$ is a parallelogram.



Problem 3

$$m_{BC} = \frac{2d}{2c-2b} = \frac{d}{c-b} \qquad \text{and} \qquad BC = \sqrt{(2c-2b)^2+(2d)^2} = 2\sqrt{(c-b)^2+d^2}$$
$$m_{ED} = \frac{d}{c-b} \qquad\qquad ED = \sqrt{(c-b)^2+d^2} = \frac{1}{2}BC$$

Therefore $m_{BC} = m_{ED}$ and $ED = \frac{1}{2}BC$ so the line segment joining the midpoints of two sides of the triangle is parallel to the third side and one-half the length of the third side.



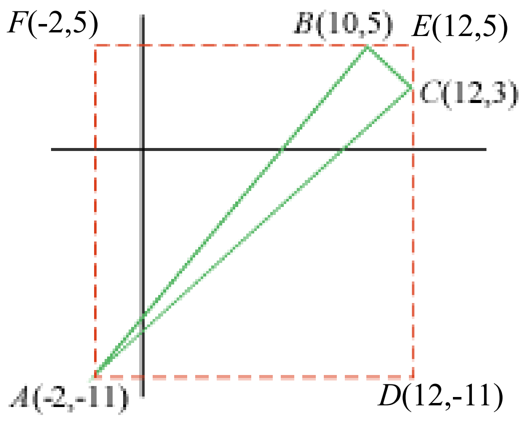
Problem 4

$A_{\triangle ABC} = A_{\text{rectangle}} - A_{\text{outer triangles}}$

$A_{\text{rectangle}} = A_{ADEF} = (12-(-2)) \times (5-(-11)) = 14 \times 16 = 224$

$A_{\text{outer triangles}} = A_{\triangle ADC} + A_{\triangle CEB} + A_{\triangle ABF} = \frac{1}{2}[14(14) + 2(2) + 16(12)] = 196$

$\therefore A_{\triangle ABC} = 224 - 196 = 28$



Verification using formula for area of a triangle:

10	5
-2	-11
12	3
10	5

$$\begin{aligned} A_{\triangle ABC} &= \frac{1}{2}\Big|10(-11)+(-2)(3)+12(5)-5(-2)-(-11)(12)-3(10)\Big| \\ &= \frac{1}{2}\Big|-110-6+60+10+132-30\Big| \\ &= 28 \end{aligned}$$

Problem 5

$$\begin{aligned} \text{Area} &= \frac{1}{2}|x_1y_2+x_2y_3+x_3y_1-x_1y_3-x_3y_2-x_2y_1| \\ &= \frac{1}{2}|(-12)(-3)+(-4)(-8)+(6)(1)-(-12)(-8)-(6)(-3)-(-4)(1)| \\ &= \frac{1}{2}|36+32+6-96+18+4| \\ &= 0 \end{aligned}$$

The triangle does not have an area because they all lie on the same line.

Problem 6

$$\begin{aligned} \text{Area} &= \frac{1}{2}|(4)(p)+(6)(6)+(0)(3)-(4)(6)-(0)(p)-(6)(3)| \\ 7 &= \frac{1}{2}|4p+36+0-24-0-18| \\ 14 &= |4p-6| \\ 7 &= |2p-3| \end{aligned}$$

Since we have an absolute value, we have two answers.

$$\begin{aligned} 7 &= 2p-3 \\ p &= 5 \end{aligned} \qquad\qquad \begin{aligned} -7 &= 2p-3 \\ p &= -2 \end{aligned}$$

Problem 7

Find point of intersection of BD and AC:

Equation of line BD:

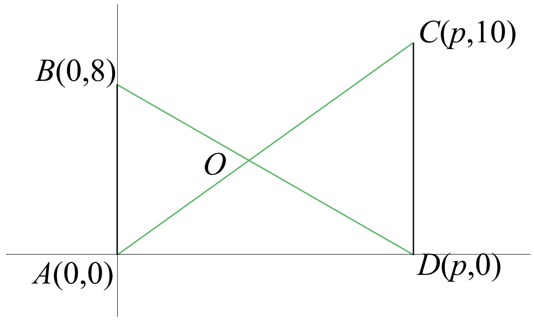
$$y = -\frac{8}{p}x + 8$$
$$x = (8-y)\frac{p}{8}$$

Equation of line AC:

$$y = \frac{10}{p}x$$
$$x = \frac{p}{10}y$$

Equating lines BD and AC:

$$\begin{aligned} (8-y)\frac{p}{8} &= \frac{p}{10}y \\ 10(8-y) &= 8y \\ 80 &= 18y \\ y &= \frac{40}{9} \end{aligned}$$



Problem 8

$$\begin{aligned}
d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\
&= \frac{|(2)(-3) + (-7)(5) + 1|}{\sqrt{2^2 + (-7)^2}} \\
&= \frac{|-6 - 35 + 1|}{\sqrt{4 + 49}} \\
&= \frac{40}{\sqrt{53}}
\end{aligned}$$

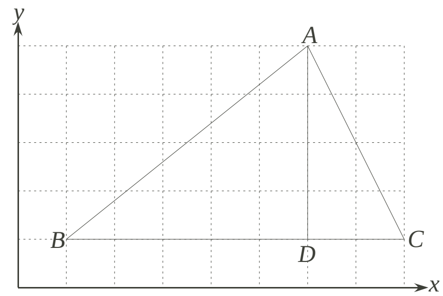
Problem 9

The slope the line is $\frac{1}{3}$, so the slope of the perpendicular is -3.

$$\begin{aligned}
-3 &= \frac{\frac{1}{3}x - \frac{2}{3} - (-6)}{x - 2} \\
-9(x-2) &= x - 2 + 18 \\
-9x + 18 &= x + 16 \\
10x &= 2 \\
x &= \frac{1}{5} \Rightarrow y = -\frac{3}{5}
\end{aligned}$$

Problem 10

Method 1:
By plotting the triangle, we can locate D.



After that, it is clear that the length of AD is 4.

Method 2:
The slope of line BC is

$$m_{BC} = \frac{1-1}{8-1} = 0$$

Since the slope of BC is 0, we know that any line perpendicular to BC is vertical. Thus $D = (6, 1)$, and the length of AD is

$$\sqrt{(6-6)^2 + (5-1)^2} = 4$$

Method 3:
Using the formula for distance from a point $(6, 5)$ to a line $y - 1 = 0$, we have

$$\begin{aligned} d &= \frac{|(0)(6) + (1)(5) + (-1)|}{\sqrt{0^2 + a^2}} \\ &= \frac{|5-1|}{1} \\ &= 4 \end{aligned}$$

Method 4:
Using the length of BC is 7. The area of the triangle is

$$\begin{aligned} \text{Area} &= \frac{1}{2}|(6)(1) + (1)(1) + (8)(5) - (6)(1) - (8)(1) - (1)(5)| \\ &= \frac{1}{2}|6 + 1 + 40 - 6 - 8 - 5| \\ \frac{1}{2}bh &= \frac{1}{2}(28) \\ 7h &= 28 \\ h &= 4 \end{aligned}$$

Problem 11

Let $P = (p, 2p{+}3)$ and $Q = (q, -q{+}2)$, The midpoint would be $M = \left(\dfrac{p+q}{2}, \dfrac{(2p+3)+(-q+2)}{2}\right)$.

$$\begin{aligned} \frac{p+q}{2} &= 2 \\ p+q &= 4 \end{aligned} \qquad\qquad \begin{aligned} \frac{(2p+3)+(-q+2)}{2} &= 5 \\ 2p-q+5 &= 10 \\ 2p-q &= 5 \end{aligned}$$

Solving these two equations, we have

$$\begin{aligned} (p+q)+(2p-q) &= 4+5 \\ 3p &= 9 \\ p &= 3 \Rightarrow q = 1 \end{aligned}$$

$\therefore P = (3, 9)$ and $Q = (1, 1)$