Intermediate Math Circles - Analytic Geometry II

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1. Summary of Analytic Geometry I

For two points $A(x_1, y_1)$ and $B(x_2, y_2)$:

- The distance, d, between the two points is calculated

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ or } d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

- The midpoint $M(x_m, y_m)$ of AB is calculated

$$M = midpoint(AB) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The slope m of AB is calculated

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(b) Special Cases Summary

(i) Slope of Horizontal Lines: $m = 0$

(ii) Slope of Vertical Lines: m is undefined

(iii) Slope of Parallel Lines: $m_1 = m_2$

(iv) Slope of Perpendicular Lines: $m_1 = \frac{-1}{m_2}$ or $m_1 \times m_2 = -1$

2. Exercises - Any Questions? Online Solutions are posted

No questions were asked concerning the problems from the previous week. Students are reminded that solutions are posted online.

3. Proofs in Analytic Geometry

(i) Prove, using analytic geometry, that the diagonals of a square right bisect each other.

The final answer here is NOT the important thing - presentation of a full and complete solution is.

The setup of an analytic proof is extremely important.

Refer to video for solution.

(ii) Prove, using analytic geometry, that the angle inscribed in a semi-circle is 90°.

The equation of a circle with radius r and centre $(0,0)$ has equation $x^2 + y^2 = r^2$.

Refer to video for solution.

4. Equations of Lines

To determine the equation of a line you need to know the slope of the line and a point on the line OR you need to know two points on the line.

In grade 9 you generally learn two forms of the equation of a line:

- Slope - Intercept Form $y = mx + b$, where m is the slope and b is the y-intercept.
- "Standard" Form $Ax + By + C = 0$ where $m = \frac{-A}{B}$, x-int$=\frac{-C}{A}$ and y-int$=\frac{-C}{B}$.

4. Equations of Lines

There are other forms of the equation of a line that you may not be familiar with.

- Point-Slope Form
  If (x_0, y_0) is a point on a line with slope m, then the equation of the line can be written $y - y_0 = m(x - x_0)$.

  If two points on the line are known, the slope can be calculated and then the above form can be used.

(iii) Determine the equation of the line through the point $A(-1, 5)$ that is perpendicular to the line $y = \frac{-3}{2}x + 5$. Express your answer in standard form.

Refer to video for solution.

- Intercept Form

  Determine the equation of the line with x-intercept a and y-intercept b.[10mm]

Refer to video for solution.

5. Summary of Different Forms of Equations of Lines

- Slope - Intercept Form $y = mx + b$, where m is the slope and b is the y-intercept.
- "Standard" Form $Ax + By + C = 0$ where $m = \frac{-A}{B}$, x-int$=\frac{-C}{A}$ and y-int$=\frac{-C}{B}$.
- Point-Slope Form $y - y_0 = m(x - x_0)$.
- Intercept Form $\frac{x}{a} + \frac{y}{b} = 1$ where a and b are the x and y intercepts, respectively.

One form of the equation may be more helpful than another as you solve problems analytically.

6. Problems Involving Equations of Lines

(iv) For the line $Ax + By + C = 0$, show that the x-int is $\frac{-C}{A}$ and the y-int is $\frac{-C}{B}$.

Refer to video for solution.

(v) Two telephone poles are 12 m apart. One pole is 8 m tall and the other is 10 m tall. A wire is strung from the top of each pole to the bottom of the other pole. The wires cross somewhere between the two poles. How high above the ground do the wires meet?

Refer to video for solution.

(vi) Determine the shortest distance from the point $P(1, 7)$ to the line $2x + y - 4 = 0$.

Refer to video for solution.

7. Distance From A Point To A Line

Since finding the distance from a point to a line is basically an algebraic problem, we can develop a general formula for the distance from a point to a line.

The distance from the point $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

We will not prove this here because of time. However, if you do a Google search on "distance of a point to a line proof" you will find an excellent proof that you could work through. Otherwise, in the Calculus Vectors course, a very efficient proof is presented.

7. Distance From A Point To A Line

We will verify our answer to the previous problem using the formula:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Find the distance from the point $P(1, 7)$ to the line $2x + y - 4 = 0$.

From the given information, $x_1 = 1$, $y_1 = 7$, $A = 2$, $B = 1$ and $C = -4$. Substituting, we obtain

$$d = \frac{|(2)(1) + (1)(7) + (-4)|}{\sqrt{(2)^2 + (1)^2}} = \frac{|2 + 7 - 4)|}{\sqrt{4 + 1}} = \frac{5}{\sqrt{5}}$$

$$d = \frac{5}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$$

This is consistent with the answer we obtained without using any formula.

8. Area of a Triangle

Determine the area of a triangle with vertices $A(3, 7)$, $B(9, 0)$ and $C(-3, -2)$ using only basic commonly known area formulas.

Refer to video for solution.

8. Area of a Triangle

Determine a formula for the area of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

This proof is the same as the previous example, be careful with the subscripts.

$$\text{Area} = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1|$$

Verify the accuracy of the formula using our previous example with vertices $A(3, 7)$, $B(9, 0)$ and $C(-3, -2)$.

Refer to video for solution.

8. Area of a Triangle

Modify the formula for the area of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ if the vertices are translated so that one vertex is at the origin.

This is left as an exercise for the students.

Verify the accuracy of the modified formula translating the vertices $A(3, 7)$, $B(9, 0)$ and $C(-3, -2)$ from our previous example so that one vertex is at the origin.

Refer to video for solution.

Problem (1)

Prove, using analytic methods, that the diagonals of a parallelogram bisect each other.

Problem (2)

For any quadrilateral $ABCD$ with P the midpoint of AB, Q the midpoint of BC, R the midpoint of CD, and S the midpoint of DA, prove, using analytic methods, that $PQRS$ is a parallelogram. (To help you get started, let A be at $(0, 0)$, B be at $(2b, 0)$, C be at $(2c, 2e)$, and D be at $(2d, 2f)$. This choice of points is helpful in finding midpoints later!)

Problem (3)

Prove, using analytic methods, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and one-half the length of the third side.

Problem (4)

Determine the area of the triangle with vertices $A(-2,-11)$, $B(10,5)$ and $C(12,3)$.

Problem (5)

Determine the area of the triangle with vertices $P(-12,1)$, $Q(-4,-3)$ and $R(6,-8)$. Interpret your result.

Problem (6)

A triangle with vertices $X(4,3)$, $Y(6,p)$ and $Z(0,6)$ has area 7 units^2. Determine all possible values for p.

Problem (7)

Two telephone poles are p units apart. One pole is 8 units tall and the other is 10 units tall. A wire is strung from the top of each pole to the bottom of the other pole. The wires cross somewhere between the two poles. How high above the ground do the wires meet? Compare this answer to the similar example done during the Math Circles.

Problem (8)

Determine the distance from the point $Q(-3, 5)$ to the line $2x - 7y + 1 = 0$.

Problem (9)

Calculate the coordinates of the foot of the perpendicular from the point $(2, -6)$ to the line $x - 3y - 2 = 0$.

Problem (10)

In $\triangle ABC$, with vertices $A(6, 5)$, $B(1, 1)$ and $C(8, 1)$, an altitude is drawn from A touching BC at D. Determine the length of the altitude AD. Develop two different solutions.

Problem (11)

A point P is chosen on the line $y = 2x + 3$ and a point Q is chosen on $y = -x + 2$. If the midpoint M of the line segment PQ is $(2, 5)$, determine the coordinates of P and Q.