UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
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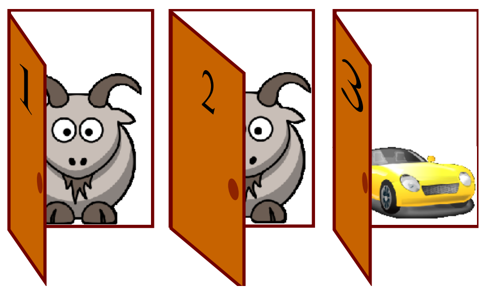
Probability I

Monty Hall Problem
You are on a game show and are given the choice of three doors. Behind one door is a car; behind the other two are goats. You pick a door, and the host, who knows what's behind the doors, opens another door, which has a goat. He asks if you would like to switch the doors or stay with the same door. Should you switch or stay? Does it matter?



You should switch. There is a probability of 1/3 that the car is in the door you choose, and a probability of 2/3 that the car is in the other door. Consider the following results if you originally choose door 1:

Door 1	Door 2	Door 3	Result if you switch	Result if you stay
Car	Goat	Goat	Goat	Car
Goat	Car	Goat	Car	Goat
Goat	Goat	Car	Car	Goat

Discrete Probability Distributions

- **Bernoulli Distribution**

The Bernoulli disttribution is a discrete distribution that has two possible outcomes, labelled by $n = 0$ and $n = 1$, in which $n = 1$ denotes a "success" and occurs with probabilty p and $n = 0$ denotes a "failure" and occurs with probability $q \equiv 1 - p$, where $0 \le p \le 1$

$$P(n) = \begin{cases} 1-p & \text{if } n = 0 \\ p & \text{if } n = 1 \end{cases}$$

Example 1 The probability of heads or tails in a coin toss is

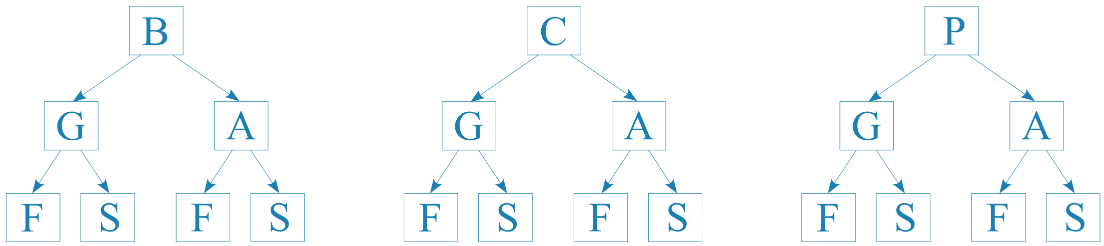
$$P(n) = \begin{cases} 1/2 & \text{if } n = \text{heads} \\ 1/2 & \text{if } n = \text{tails} \end{cases}$$

Note: $P(0) + P(1) = 1$

Review: Counting

Fundamental Counting Principle of Multiplication
If one event can occur in m ways and a second event can occur in n ways, then both events can occur in $m \times n$ ways, provided the outcome of the first event does not influence the outcome of the second event.

Example 2 Katy is out for dinner and she needs to chose what she wants to eat. She needs one meat portion, one vegetable portion and one side. For her meat portion, she can choose beef, chicken or pork; for her vegetable portion she can choose either green beans or asparagus; and for her side she can choose fries or a salad. How many different ways can she make her meal?



All the ways Katy can make her meal is:

- (Beef, Green beans, Fries), (Beef, Green beans, Salad), (Beef, Asparagus, Fries), (Beef, Asparagus, Salad)
- (Chicken, Green beans, Fries), (Chicken, Green beans, Salad), (Chicken, Asparagus, Fries), (Chicken, Asparagus, Salad)
- (Pork, Green beans, Fries), (Pork, Green beans, Salad), (Pork, Asparagus, Fries), (Pork, Asparagus, Salad)

For a total of 12 ways

- First event is (beef, chicken, pork). There are 3 ways in which this can be accomplished.
- Second event is (green beans, asparagus). There are 2 ways in which this can be accomplished.
- Third event is (fries, salade). There are 2 ways in which this can be accomplished.

Using the fundamental counting principle of multiplication we get:

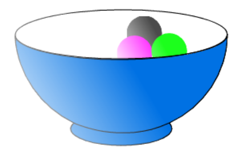
$$\begin{aligned}\text{Total number of ways to make her meal} &= 3 \times 2 \times 2 \\ &= 12\end{aligned}$$

Example 3 How many 3-digit numbers can you make with the 4 digits: 1, 2, 3, 4, if:

a) You can repeat digits in the numbers.
b) No digit can appear more than once in the number.
c) The number is even and no digit can appear more than once in the number.
d) The digit 3 must be included in the number and no digit can appear more than once in the number.

a) $4 \times 4 \times 4 = 4^3 = 64$
b) $4 \times 3 \times 2 = 24$
c) $3 \times 2 \times 2 = 12$
d) $3 \times (1 \times 3 \times 2) = 18$

Example 4 Find the total number of ways to select 2 marbles out of a bowl of 3 marbles, where the order you choose the marbles does not matter.



There are 3 ways to pick the first item and 2 ways to pick the second item. This gives us:

$$\boxed{3} \times \boxed{2} = \boxed{6}$$

This means there are 6 ways of pulling the objects out: (1,2), (1,3), (2,1), (2,3), (3,1), (3,2). However, if the order does not matter, we can see that we have to eliminate the reorderings, ex: (1,2) is the same as (2,1)
In this case there are 2 ways of reordering each unique set. Therefore there are $6 \div 2 = 3$ ways to select 2 out of 3 items, where order does not matter.

In general if we are picking n items, there are

$$n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

ways of arranging each unique combination.

Factorial n!, read as "n factorial", is defined as the product of positive integers less than or equal to n:

$$n! := n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

Note: $0! = 1$

Choose $\binom{n}{r}$, read as "n choose r", counts the number of distinct subsets of $\{1, 2, 3, ..., n-1, n\}$ having exactly r elements:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

We use this when we want to count the number of distinct ways we can pick r objects, when there are a total of n objects to choose from. Repeat the example above using the definition of choose. Do you get the same answer?

Example 5 How many distinct five-letter words can you spell by rearranging the letters in the word "apple"?

The number of ways to rearrange the five-letters is 5!, however, since there are 2 "p's" there is going to be an overlap in words. (ex: ap_1lep_2 and ap_2lep_1). So since there are 2! ways to order the "p's", the number of distinct words are: $5! \div 2!$

Exercise 1 If two heart cards, two club cards, and three spade cards are taken from a deck of cards, how many ways can you choose 5 cards such that you choose one heart, two clubs and 2 spades?

Exercise 2 How many distinct 7-digit numbers can you make by rearranging the digits of 2335142? **Challenge:** How many 5-digit numbers can you make from the same 7 numbers?

Problem Set

1. After months of begging, your parents finally agree to get you a cell phone there are two plans that you have the choice of getting, in each plan you have to choose one of unlimited texting, unlimited evenings and weekends, or my five (unlimited talk and text to your five favourite numbers), and your parents are letting you get one out of five add-ons that they chose. How many different ways are there for you to set up your plan?

2. In a deck of 52 cards, how many ways can you choose 5 cards having **at least** 2 kings?

3. Nine people enter a 100-meter dash. Assuming no ties, how many ways can the gold, silver, and bronze metals be awarded?

4. In a spelling bee prep course, there are 20 contestant and 10 instructors. How many ways can an elite group be formed of 3 contestants, and 2 instructors?

5. Your school wants to make a student councel. The students who wish to be on the councel can be chosen for any one position: President, Vice-President, Secretary, and one of two Treasurers. How many ways can the councel be chosen if:
   a) 5 students want to be on the councel?
   b) 30 students want to be on the councel?

6. How many ways can Sarah arrange 7 different place settings for her family dinner so that she is sitting in the fifth spot? If Sarah want to sit next to her twin sister Sally how many ways can the seats be arranged?

7. Three cards are chosen at random from a deck without replacement. What is the probability of getting a jack, a queen and a king?

8. Two dice are rolled. What is the probability of rolling a 6 between the two die.

9. Christina has a pack of 10 different coloured pens: black, grey, brown, green, blue, purple, pink, yellow, orange and red. She needs to choose 6 colours of pens for her drawing. She chooses at most one of black, grey, or brown, and any of the others. How many different ways can she choose her pens?