Math Circles:
Invariants Part 1

March 23, 2011

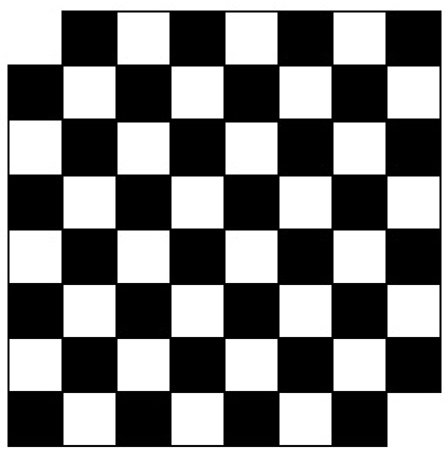
What is an Invariant?

In the simplest terms, an invariant is something that doesn't change. As easy as that sounds, invariance is a very powerful property that is used widely in contests. Using invariance, we can tackle problems with games, colorings, symmetry, parity, sometimes even induction. An invariance problem can come in many different shapes and forms - the trickiest part is recognizing it.

Many coloring problems are invariants in disguise. Grab your favorite colors, and let's look at these examples.

Example 1: A Classic

You start off with an 8×8 chessboard. For some reason, your chessboard is missing two opposite corners, as shown below. You have 31 dominoes, and you want to tile what's left of the chessboard with these dominoes. (A domino is a 2×1 tile) Can you do it?



Solution:

After trying yourself to tile the chessboard with the dominoes, we can quickly see that it seems impossible. But how do we prove this? Well, let's look at the number of black and white squares on the board. In this case, we have 32 black squares, and 30 white squares. However, a dominoe, in any orientation, will cover exactly 1 white square and 1 black square. This means that to cover the board perfectly, we need an equal number of black and white squares - which we don't have. Therefore, it will be impossible for us to cover our chessboard with these dominoes if the corner pieces are missing.

I mentioned earlier that we can use the idea of invariance to play games. Here's another fun problem - I think this was actually from an old Euclid.

Example 2: Marbles

There are 5 red marbles and 6 green marbles in a jar. Pascal plays a strange game. He removes two marbles at a time, with the following rules:

1. If the marbles are both green, he puts one green marble back.
2. If there is one marble of each colour, he puts one red marble back.
3. If the marbles are both red, he puts one green marble back.

At the end, there will be one marble left. Which colour is it?

Solution:

Well, let's consider each of the three possible scenarios.

1. Both marbles are green: Since we put one back in this case, red marbles are unchanged, and there's -1 green marbles
2. One red and one green marble: In this case the red marble gets put back, and so we have unchanged red marbles, and -1 green marble
3. Both marbles are red: We put one green marble back. Now there's -2 red marbles, and +1 green marbles.

So in this case, what's the invariant? Well, notice the red marbles change by an even number each time, and the green marbles change by either +1 or -1. Remember we started with 5 red marbles, and 6 green ones. If the red marbles can only decrease by 0 or 2 - notice we will always have an odd number of red marbles. Green marbles can be either even or odd numbered. Thus, if there's one marble left - it must be red!

Another area we can see invariants is in various Chessboard problems. Here's another example.

Example 3: A chessboard game

Hypatia and Gauss are going to play a game with the following rules.

1. Hypatia begins by placing a knight on any square of a regular 8×8 chessboard
2. Gauss moves the knight first
3. Hypatia and Gauss alternate moving the knight, but they can only move the knight to squares that it has never been to before
4. The player who cannot move the knight anymore loses

A knight is a chess piece with an interesting form of movement. It can either move one square vertically and two horizontally, or it can move one square horizontally and two squares vertically.

It's movement looks like an L shape.

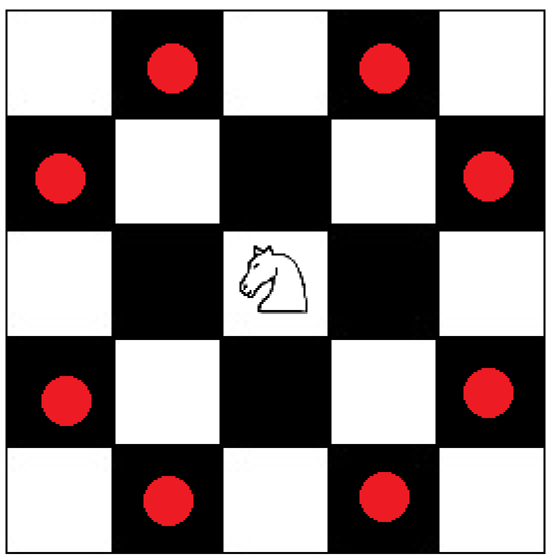
With the correct strategy and perfect play, either Hypatia or Gauss will always be able to win. Who has the winning strategy? Explain!!

<u>*Solution:*</u>

When dealing with this type of problem - the first thing to do is just try it. Grab a friend or do it on your own, get a feel for the game play. This is the first way to build intuition for finding a winning strategy.

The next step, after playing a few times, is trying to figure out who wins. To start, let's look at how a knight moves.

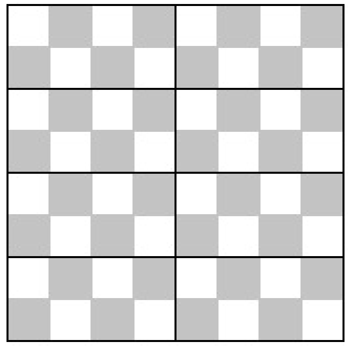
Since it is on a chessboard, the piece will start on either a black square, or a white square. Suppose it starts on a white square. Looking closely at all the different paths it can take below, it will always end up on a black square. Similarly, a knight starting on a black square will always end up on a white square after it is moved.



Suppose that player 1 starts with the knight on a white square. Then player 2 will move the knight from white to black, ending on a black square. Then player 1 will move the knight from black to white, ending on a white square.

This means that Player 1's turn will always end on a white square, and Player 2's turn will always end on a black square. On the chessboard, there are 32 white squares, and 32 black squares. Theoretically, if every square on the board gets used in this game then, the last square on the board the knight sits on will have to be a black square then. Since Player 2's turn ends on a black square, it makes sense that Player 2 should always win.

Can we come up with a winning strategy? Consider dividing up the board as shown below, in 8 2×4 blocks.



Let's look at each 2×4 block. Player 1 will start off in one of these blocks. Notice that Player 2 has exactly 1 move that moves the knight - but stays inside that block.

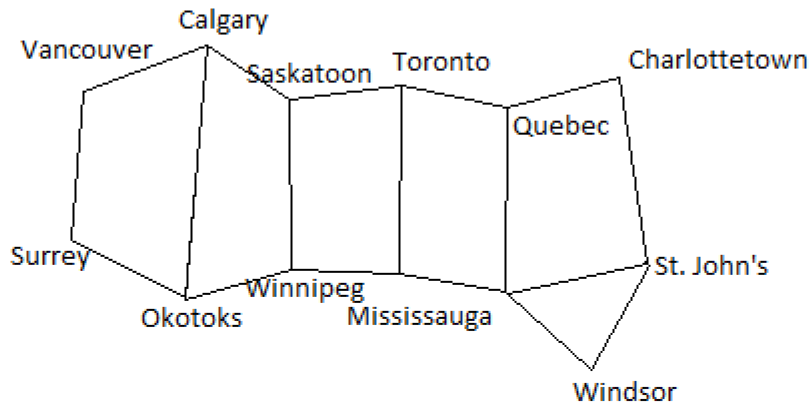
After Player 2 has moved, Player 1's only option is to move the knight outside of this 2×4 block into another one. What does this mean? It means that no matter what Player 1 does, there will always be a move for Player 2 - by taking staying inside the rectangle. By following this strategy, Player 1 will eventually run out of moves, allowing Player 2 to win.

In that question, the invariance that was key was noticing how a knight changes colors when moving, and that Player 1 will always end on one color, and Player 2 on the other. Combined with a clever trick in dividing up the board, you now have a game you can always win if you play against your friends!!

In most of the examples above, we found invariants that don't change. However, sometimes we get an invariant that does change - but it's an invariant because we can predict the change. Consider our last example:

Example 4: Cross-country race

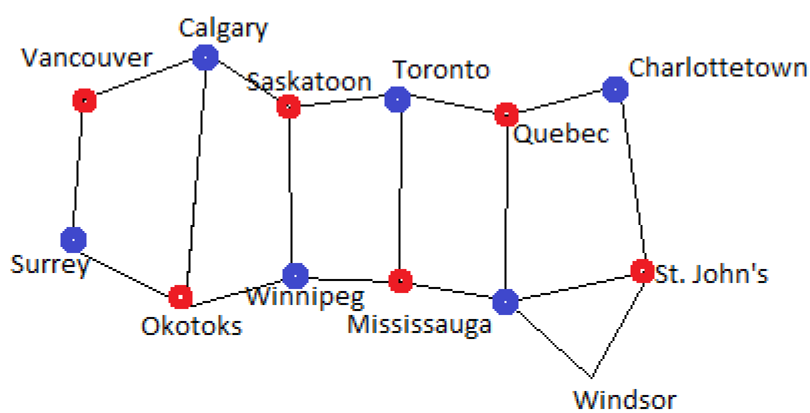
Flynn has stolen a million dollars from Maximus, and has run away with it. Maximus has discovered this, and wants to get the money back. Maximus starts in Okotoks, and Flynn starts in Vancouver. Because of road construction, it is only possible to travel between certain cities, as indicated in the diagram below. They take turns moving, and Maximus gets his money back by moving to the city that Flynn is at. Maximus moves first. Can Maximus get his million dollars back? If so, how? If not, explain why.



Solution:

This problem uses a similar idea to the last one. Though we're not dealing with a checkerboard, we can still color the cities in a way that makes this problem easier.

Starting with Vancouver, color the cities red and blue in an alternating pattern. Any two cities that are connected by a road should have different colors - except for the one city Windsor. For Windsor, leave that city uncolored.



Next, explore the type of moves that each player can do. Player 1, Maximus, starts in a red city. On each of his turns then, he can move to a blue city. On the other hand, Player 2, Flynn, is also starting in a red city, and will move to a blue city after Maximus moves. If Maximus is in a blue city then, he will move to a red city, while Flynn is in a blue city.

For Maximus to catch Flynn, he needs to end up in the same color city as Flynn is. Right now, this is impossible! For Maximus to catch Flynn - he needs to somehow change the color he ends up on - can he do that?

Turns out, he can. By going to Windsor, he can actually change the order of colors he ends up on, allowing him to end on the same color that Flynn is on after a move. Try it, and find out! The problem just reduces to cornering Flynn in some corner of the country, and Maximus will eventually get his money back.

To end off, here's one more problem.

Example 4: Negative or Positive?

In the figure below, you may switch the signs from all the numbers in one row, one column, or a parallel to one of the diagonals. In specific, you may switch the sign of any of the corner squares. Prove that there must always be at least one -1 in the table.

1	1	1	1
1	1	1	1
1	1	1	1
1	1	-1	1

Solution:

Notice that the product of all 8 boundary squares (excluding the corners), is invariant under each move. It begins and ends at -1 - so there will always be at least one -1 in the table.

Invariance Introduction

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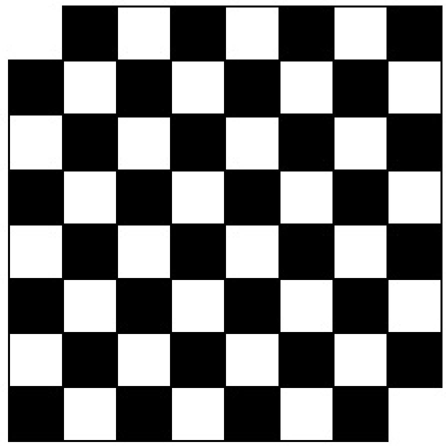

2. There are 5 red marbles and 6 green marbles in a jar. Pascal plays a strange game. He removes two marbles at a time, with the following rules:

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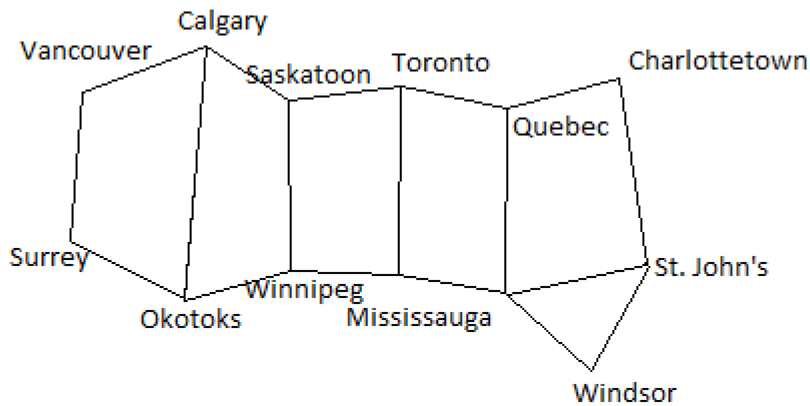

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1	1	1	1
1	1	-1	1

Invariance Problem Set

1. Sarah writes the numbers $1, 2, \ldots, 2n$ for an odd integer n, on the board. She performs the following operation: she picks two numbers at random from the board (call them a and b), erases them, and writes down $|a - b|$. She continues to do this until there is only one number left at the end. Prove that an odd number will remain at the end.

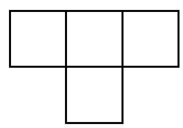
2. There are 13 green, 15 blue, and 17 red chameleons on Camelot Island. Whenever two chameleons of different colors meet, they both swap to the third color (i.e., a green and blue would both become red). Is it possible for all chameleons to become one color?

3. An $m \times n$ table is filled with numbers such that each row and column sum up to 1. Prove that $m = n$.

4. Assume an 8×8 chessboard, in the usual coloring. You can repaint all the squares of a row or column, i.e., all white squares become black, and all white squares become black. Can you get exactly one black square?

5. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed, but we have one more tile of the other kind available. Can we retile the floor perfectly?

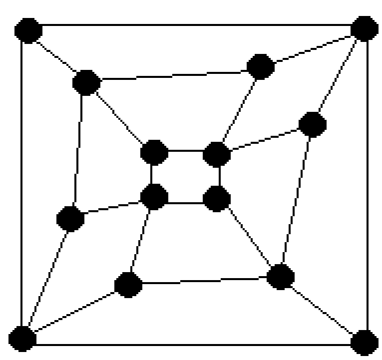
6. Prove that a 10×10 board cannot be covered by T-shaped tetrominos (shown below).



7. Show that an 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino (a 2×2 tile).

8. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row, or subtract 1 from each number of a column. Prove that you can reach a table of zeros by using only these moves.

9. The figure shows a map of 14 cities. Can you follow a path that passes through each city exactly once?



10. A hockey player has 3 pucks labelled A, B, C in an arena (as in all math, an infinite area!). He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 2011 hits?