# Intermediate Math Circles
# November 09, 2011
# Counting III

Last time, we looked at combinations and saw that we still need to use the product and sum rule to solve many of the problems. Today, we start by looking at how to count the number of permutations when the objects are not all distinct. We will then look at permutations and combinations when repetitions are allowed.

Example 1:

Calculate the number of possible permutations on the set $\{A, A, B, B, B\}$.

Solution:

Let's first list all the possibilities and then try to figure out how we could have calculated the result mathematically. Using a systematic approach, we get

A in the first position: $AABBB,\ ABABB,\ ABBAB,\ ABBBA$
A in the second position: $BAABB,\ BABAB,\ BABBA$
A in the third position: $BBAAB, BBABA$
A in the fourth position: $BBBAA$

There are 10 possible ways to permute the set $\{A, A, B, B, B\}$.

It seems that in our writing of the possibilities we were looking at how to place the $A$'s. Once the $A$'s were placed, the $B$'s had to go in the remaining spots.

Consider that we have 5 spots and we have to choose 2 of them for the $A$'s. We can do this in $\binom{5}{2}$ ways. Once the $A$'s are placed there is only 1 way to place the $B$'s. $\therefore$ there are $\binom{5}{2} \times 1$ or 10 ways to permute the letters.

Of course, we could have instead counted the number of ways we could have placed the 3 $B$'s into the 5 spots giving us $\binom{5}{3}$ or 10 ways as before.

Example 2:

Calculate the number of possible permutations on a set with $m$ $A$'s and $n$ $B$'s.

Solution:

We have $m + n$ spots. We need to choose $m$ of them for the $A$'s so we have $\binom{m+n}{m}$ ways of placing the $A$'s. Once the $A$'s are placed the $B$'s must go in the remaining $n$ spots in 1 way.

Again, we could have chosen $n$ spots from $m + n$ spots to place the $B$'s. This would result in $\binom{m+n}{n}$ ways of permuting the letters. It is left as an exercise for the reader to show that the two expressions are equal.

Example 3:

Calculate the number of permutations on a set with 3 $A$'s, 6 $B$'s and 4 $C$'s.

Solution:

We now see that we have 13 possible positions. We can place the 3 A's first in $\binom{13}{3}$ ways. Once the A's are placed there are 10 spots left and we can place the B's in $\binom{10}{6}$ ways. Once the A's and B's are placed there are 4 spots left for the 4 C's. The C's can only be placed in 1 way. The number of ways to permute the letters is

$$\binom{13}{3} \times \binom{10}{6} \times 1 = \frac{13!}{3! \times 10!} \times \frac{10!}{6! \times 4!} = \frac{13!}{3! \times 6! \times 4!} = 60\ 060.$$

There are 60 060 ways to arrange the letters.

Does it matter which order we place the letters? If we place the $B$'s first, there are $\binom{13}{6}$ ways to place them. Place the $C$'s next by choosing 4 of the remaining 7 spots in $\binom{7}{4}$ ways. The 3 $A$'s can now only be placed 1 way in the 3 remaining spots so the number of ways to permute the letters is

$$\binom{13}{6} \times \binom{7}{4} \times 1 = \frac{13!}{7! \times 6!} \times \frac{7!}{3! \times 4!} = \frac{13!}{6! \times 3! \times 4!} = \frac{13!}{3! \times 6! \times 4!}, \text{ as before.}$$

Example 4:

Calculate the number of permutations on a set with 3 $A$'s, 6 $B$'s and 4 $C$'s such that the permutation must start with an $A$ and end with a $C$.

Solution:

There is only one way to start a permutation with an $A$, put an $A$ in the first place leaving 2 $A$'s to arrange later. There is only one way to end a permutation with a $C$, put a $C$ in the last place leaving 3 $C$'s to arrange later. We now see that we have 11 positions to fill. We can place the remaining 2 A's first in $\binom{11}{2}$ ways. Once the A's are placed there are 9 spots left and we can place the B's in $\binom{9}{6}$ ways. Once the A's and B's are placed there are 3 spots left for the 3 remaining C's. The remaining C's can only be placed in 1 way. The number of ways to permute the letters is

$$1 \times 1 \times \binom{11}{2} \times \binom{9}{6} \times 1 = \frac{11!}{9! \times 2!} \times \frac{9!}{3! \times 6!} = \frac{11!}{2! \times 3! \times 6!} = 4\ 620.$$

There are 4 620 ways to arrange the letters.

## Exercises

1. How many permutations are there on a set with 2 $A$'s, 4 $B$'s, 3 $C$'s, 2 $D$'s.

2. How many permutations are there on the letters of "CALCULATE".

3. How many numbers (using all of the digits) can be formed from a set with 5 ones, 4 twos and 5 threes such that:
   (a) there are no restrictions.

   (b) the number must be even.

   (c) the number must be odd.

   (d) the number one must always be followed immediately by a three.

4. How many permutations are there on the letters of "LETTERS" that start with an $R$?

5. How many permutations are there on the letters of $INTERMEDIATE$ such that:
   (a) there are no restrictions.

   (b) the permutation must start with the letter $I$ and end with the letter $E$.

   (c) the $I$'s must be together, the $T$'s must be together and the $E$'s must be together.

   (d) the $N$ must never immediately follow an $I$.

**Answers:**

1. $\binom{11}{2} \times \binom{9}{4} \times \binom{5}{3} \times 1 = \frac{11!}{9!\times 2!} \times \frac{9!}{5!\times 4!} \times \frac{5!}{2!\times 3!} = \frac{11!}{2!\times 4!\times 2!\times 3!} = 69\ 300$

2. $\binom{9}{2} \times \binom{7}{2} \times \binom{5}{2} \times 3! = \frac{9!}{7!\times 2!} \times \frac{7!}{5!\times 2!} \times \frac{5!}{3!\times 2!} = \frac{9!}{2!\times 2!\times 2!} = 45\ 360$

3. a) $\binom{14}{5} \times \binom{9}{4} \times 1 = \frac{14!}{9!\times 5!} \times \frac{9!}{5!\times 4!} = \frac{14!}{5!\times 5!\times 4!} = 252\ 252$

   b) $1 \times 1 \times \binom{13}{5} \times \binom{8}{5} = \frac{13!}{8!\times 5!} \times \frac{8!}{3!\times 5!} = \frac{13!}{5!\times 5!\times 3!} = 72\ 072$

   c) $252\ 252 - 72\ 072 = 180\ 180$

   d) $\binom{9}{5} = \frac{9!}{4!\times 5!} = 126$

4. $1 \times \binom{6}{2} \times \binom{4}{2} \times 2! = 1 \times \frac{6!}{2!\times 2!} = 180$

5. a) $\binom{12}{3} \times \binom{9}{2} \times \binom{7}{2} \times 5! = \frac{12!}{9!\times 3!} \times \frac{9!}{7!\times 2!} \times \frac{7!}{5!\times 2!} \times 5! = \frac{12!}{3!\times 2!\times 2!} = 19\ 958\ 400$

   b) $1 \times 1 \times \binom{10}{2} \times \binom{8}{2} \times 6! = \frac{10!}{8!\times 2!} \times \frac{8!}{6!\times 2!} \times 6! = \frac{10!}{2!\times 2!} = 907\ 200$

   c) $8! = 40\ 320$

   d) $19\ 958\ 400 - \binom{11}{2} \times \binom{9}{2} \times \binom{7}{3} \times 4! = 19\ 958\ 400 - \frac{11!}{2!\times 2!\times 3!} = 18\ 295\ 200$

---

## Permutations with Repetition

We now look at permutations with repetition allowed. That is how many ways can we choose $r$ objects out of $n$ objects when we are allowed to select any of the $n$ objects more than once.

Example 5:

Anna Lize gets to pick a 4-digit combination for her combination lock. How many different combinations are possible?

Solution:

This is a permutation with repetition problem since she needs to pick 4 digits from the set $\{0, 1, \ldots, 9\}$. However, we also see that this is just a simple product rule problem. In each case she has 10 choices for each digit, so the total number of combinations is $10\times 10\times 10\times 10\times 10 = 10^4$.

It is relatively easy to see that permutations with repetitions are just product rule problems. There are always $n^r$ ways of choosing $r$ objects out of $n$ objects when repetition is allowed. Thus, we move onto the more interesting problem of combinations with repetition.

## Combinations with Repetition

Example 6:

A pop machine offers 6 kinds of pop. Polly Knowmeal wants to purchase 4 cans. How many different purchases can she make?

Solution:

For our first attempt we will break the problem into distinct cases so that it is possible to use the sum rule.

Case 1: Polly purchases 4 cans of the same type of pop. She can do this in $\binom{6}{1} = 6$ ways.

Case 2: Polly purchases 3 cans of the same type of pop and one can of a different type. She can do this in $\binom{6}{1} \times \binom{5}{1} = 6 \times 5 = 30$ ways.

Case 3: Polly purchases 2 cans of the one type of pop and two cans of another type.. She can do this in $\binom{6}{2} = 15$ ways.

Case 4: Polly purchases 2 cans of the one type of pop and then selects one can of each of two other kinds of pop. She can do this in $\binom{6}{1} \times \binom{5}{2} = 6 \times 10 = 60$ ways.

Case 5: Polly purchases 1 can of each of four different types of pop. She can do this in $\binom{6}{4} = 15$ ways.

Using the sum rule, Polly has $6 + 30 + 15 + 60 + 15 = 126$ ways to make her selection.

This method has a few drawbacks. You may not think of all the possible cases. Your cases may not all be disjoint. We will look at another way to count situations like this in two weeks.