Intermediate Math Circles
November 02, 2011
Counting II

Last week, after looking at the product and sum rules, we looked at counting permutations of objects. We first counted permutations of entire sets and ended our night by looking at the number of ways we could choose k objects from n distinct objects where the order we selected the objects was important.

Today, we will look at ***combinations***: the number of ways in which we can choose k objects from n distinct objects, where the order of selection does not matter.

Example 1:

A math student is given a list of 5 math problems and is asked to do solve any 3 of the problems. How many different problem selections can the student make?

Solution:

Let us name the problems 1, 2, 3, 4, and 5. Then, the possible choices are:

$$\begin{array}{ccccc} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,3,4\} & \{1,3,5\} \\ \{1,4,5\} & \{2,3,4\} & \{2,3,5\} & \{2,4,5\} & \{3,4,5\} \end{array}$$

Observe that since it does not matter which order the student does the problems that to list the choice $\{1, 2, 3\}$ would be the same as listing the choices $\{3, 1, 2\}$ or $\{2, 3, 1\}$, etc.

We want to figure out how to count these mathematically. One way we can think about this is that we want to count all the permutations of the n objects taken k at a time, and then remove all the permutations which are the same.

In the example above, the total number of permutations is $5 \times 4 \times 3 = 60$.

For any particular permutation, how many other permutations contain exactly the same numbers? Consider the permutation $\{(1, 2, 3)\}$. How many other permutations contain exactly the same three numbers? This would be all the permutations on 3 objects so it is $3! = 6$.

So we have 60 total permutations, but we can organize these into groups of 6, which all contain the same objects. Since order does not matter, the total number of possible choices is 60 objects divided by 6 groups = 10. This matches what we did above when we listed all of the possibilities.

Therefore, the number of ways in which we can choose k objects from n distinct objects when order does not matter is

$$\frac{n \times (n-1) \times \cdots \times (n-(k-1))}{k!}$$

In our example, $n = 5$ and $k + 3$ so

$$\frac{n \times (n-1) \times \cdots \times (n-(k-1))}{k!} = \frac{5 \times (5-1) \times (5-(3-1))}{3!} = \frac{5 \times (4) \times (3)}{6} = 10.$$

The formula does **not** look very nice in this form, so we proceed to make it look prettier.

Observe that $5 \times 4 \times 3 = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \dfrac{5!}{2!} = \dfrac{5!}{(5-3)!}$.

In general, $n \times (n-1) \times \cdots \times (n-(k-1)) = \dfrac{n!}{(n-k)!}$.

Thus, we can write the number of permutations on n objects taken k at a time as $\dfrac{n!}{(n-k)!}$.

Then the number of ways to choose k objects from n distinct objects where order is not important is the number of permutations divided by the number of ways of arranging k objects. That is,

$$\frac{n!}{(n-k)!} \div k! = \frac{n!}{(n-k)!k!}.$$

We denote this with the symbol $\dbinom{n}{k}$ and say "n choose k". $\therefore \dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$

Example 2:

At a cafeteria, a student is allowed to pick 4 different items from the following list: pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, and apple pie.

a.) How many different choices does the student have?

Solution: The student can choose 4 items from 10 items.

$$\binom{10}{4} = \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times \cancel{8} \times 7}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 10 \times 3 \times 7 = 210.$$

$\therefore$The student has 210 choices for his meal.

b) How many different choices does the student have, if they don't like apple pie?

Solution: Now the student can choose 4 items from only 9 items.

$$\binom{9}{4} = \frac{9!}{5! \times 4!} = \frac{9 \times \cancel{8} \times 7 \times 6}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 3 \times 7 \times 6 = 126.$$

$\therefore$The student has 126 choices for his meal.

c) How many different choice does the student have, if they must pick one and only one drink?

Solution: The student has 4 choices of drink to select from. Then, the student can pick any 3 of the remaining 6 non-drink items so has $\dbinom{6}{3}$ ways of picking those. By the product rule, the student has $4 \times \dbinom{6}{3} = 4 \times \dfrac{\cancel{6} \times 5 \times 4}{\cancel{3} \times \cancel{2} \times 1} = 4 \times 5 \times 4 = 80$ choices.

Exercises

1. Evaluate the following:

   a.) $\binom{7}{3}$ b.) $\binom{10}{2}$ c.) $\dfrac{\binom{6}{4}}{\binom{6}{2}}$

2. For a quest, the knight, Sir Cumference, needs to pick 4 out of his 10 fellow knights.

   a.) In how many ways can he do this?

   b.) In how many ways can he do this, if he must pick one particular knight, Sir Kull.

   c.) In how many ways can he do this, if he can't pick Sir Kull?

3. A student, taking a math test, is told to answer any 7 of the 10 questions. In how many ways can the student do this?

4. A high-school of 520 boys and 480 girls must select 5 students to represent them at a competition.

   a.) In how many ways can they do this?

   b.) In how many ways can they do this if boys are not allowed to be selected?

   c.) In how many ways can they do this if the group must have 3 boys and 2 girls?

   d.) In how many ways can they do this if the group must have more boys than girls?

Answers:

1. a) 35 b) 45 c) 1

2. a) $\binom{10}{4} = 210$ b) $\binom{9}{3} = 84$ c) $\binom{9}{4} = 126$

3. $\binom{10}{7} = 120$

4. a) $\binom{1000}{5}$ b) $\binom{480}{5}$ c) $\binom{520}{3} \times \binom{480}{2}$

4. d) $\binom{520}{5} + \binom{520}{4} \times \binom{480}{1} + \binom{520}{3} \times \binom{480}{2}$

By examining the values of $\binom{n}{k}$ as we change n and k, a pattern emerges. Together, we will proceed to find the pattern.

We begin with $\binom{0}{0}$. This is the number of ways we can pick 0 objects from 0 objects. We are not really doing much so we will say there is just 1 way of doing this. So $\binom{0}{0} = 1$.

What is $\binom{1}{0}$? How many ways can we pick 0 objects from 1 object? There is only one way, we simply don't pick any objects. So $\binom{1}{0} = 1$. Notice then that we have $\binom{1}{0} = \frac{1!}{0! \times 1!} = 1$, so we define $0! = 1$.

What is $\binom{1}{1}$? How many ways can we pick 1 object from 1 object? There is only one way, we pick the object. So $\binom{1}{1} = 1$. Notice that we have $\binom{1}{1} = \frac{1!}{1! \times 0!} = 1$, so again we see that $0!$ must equal 1.

We calculate that $\binom{2}{0} = \frac{2!}{2! \times 0!} = 1$, $\binom{2}{1} = \frac{2!}{1! \times 1!} = 2$, $\binom{2}{2} = \frac{2!}{0! \times 2!} = 1$,
$\binom{3}{0} = \frac{3!}{3! \times 0!} = 1$, $\binom{3}{1} = \frac{3!}{2! \times 1!} = 3$, $\binom{3}{2} = \frac{3!}{1! \times 2!} = 3$, $\binom{3}{3} = \frac{3!}{0! \times 3!} = 1$.

Let's start making these into a table in the form of a triangle:

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Substituting in the values gives

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

Exercise: Fill in the next 3 rows in the above triangle. Try to find patterns in the triangle and use it to fill in 3 more rows.

Answer:

Observe that the ends of each row are always 1 and the middle numbers are just the sum of the two numbers above it. This gives the triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

which is called **Pascal's triangle**.

Pascal's triangle has many interesting properties and uses. One immediate use is that it suggests some properties of $\binom{n}{k}$.

In particular, notice that $\binom{n}{k} = \binom{n}{n-k}$. But Pascal's triangle is not a proof of this fact, So let us prove it.

Proposition: For any non-negative integers n and k with $n \geq k$ we have $\binom{n}{k} = \binom{n}{n-k}$.

Proof:

We have $\binom{n}{n-k} = \dfrac{n!}{(n-(n-k))! \times (n-k)!} = \dfrac{n!}{k! \times (n-k)!} = \dfrac{n!}{(n-k)! \times k!} = \binom{n}{k}$. $\square$

The triangle also suggests another property which is much harder to prove, namely that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

We will not prove this proposition here but will illustrate it by referring back to Exercise 1 (a): For a quest, the knight, Sir Cumference, needs to pick 4 out of his 10 fellow knights. In how many ways can he do this?

The number of ways to choose the knights was $\binom{10}{4}$.

We could have solved the problem by counting the number of groups with Sir Kull in it and adding the number of groups without Sir Kull in it.

Using the sum rule we obtain $\binom{9}{3} + \binom{9}{4}$.

$$\therefore \binom{10}{4} = \binom{9}{3} + \binom{9}{4}.$$

Exercises

1. **a.)** Mr E. Lipps has in front of him a circle, a square, and a rectangle. In how many ways can he select some of the shapes? (He may pick any number of the shapes.)

   b.) If he has 4 shapes instead of three, in how many ways can he select some of the shapes?

   c.) What if he had n shapes?

2. How many games are held in a round-robin singles tennis tournament involving n players where every player plays every other player once?

3. Determine $\dfrac{\binom{n}{k}}{\binom{n}{k-1}}$.

Answers:

1. a) $\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=3+3+1=7=2^3-1$

1. b) $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}=4+6+4+1=15=2^4-1$

1. c) 2^n-1

2. $\binom{n}{2}=\dfrac{n(n-1)}{2}$

3. $\dfrac{n-k+1}{k}$