Some practice questions for CIMC.

1. Determine the value of $\frac{\sqrt{25-16}}{\sqrt{25}-\sqrt{16}}$.

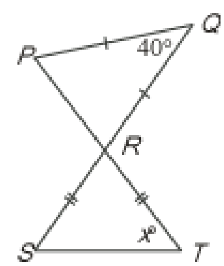
   {2008 Cayley #2}

   Solution

$$\begin{aligned}\frac{\sqrt{25-16}}{\sqrt{25}-\sqrt{16}} &= \frac{\sqrt{9}}{5-4}\\ &= \frac{3}{1}\\ &= 3\end{aligned}$$

2. In the diagram, PT and QS are straight lines intersecting at R such that $QP = QR$ and $RS = RT$. Determine the value of x.

   {2008 Cayley #8}



   Solution
   Since $PQ = QR$, then $\angle QPR = \angle QRP$.
   Since $\angle PQR + \angle QPR + \angle QRP = 180°$, then $40° + 2(\angle QRP) = 180°$, so $2(\angle QRP) = 140°$ or $\angle QRP = 70°$.
   Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 70°$.
   Since $RS = RT$, then $\angle RST = \angle RTS = x°$.
   Since $\angle SRT + \angle RST + \angle RTS = 180°$, then $70° + 2x° = 180°$ or $2x° = 110°$ or $x = 55$.

3. If $x + y + z = 25$, $x + y = 19$ and $y + z = 18$, determine the value of y.

   {1998 Cayley #11}

   Solution
   We are given that

$$\begin{aligned}x + y + z &= 25 \quad (1)\\ x + y &= 19 \quad (2)\\ y + z &= 18 \quad (3)\end{aligned}$$

   Add equations (2) and (3) to get $x + 2y + z = 37$ (4). Subtraction equation (1) from eqution (4) to get $y = 12$.

4. The odd numbers from 5 to 21 are used to build a 3 by 3 magic square. (In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum.) If 5, 9 and 17 are placed as shown, what is the value of x?

	5	
9		17
x		

{2010 Cayley #16}

Solution

The sum of the odd numbers from 5 to 21 is

$$5+7+9+11+13+15+17+19+21=117$$

Therefore, the sum of the numbers in any row is one-third of this total, or 39.
This means as well that the sum of the numbers in any column or diagonal is also 39.
Since the numbers in the middle row add to 39, then the number in the centre square is $39 - 9 - 17 = 13$.
Since the numbers in the middle column add to 39, then the number in the middle square in the bottom row is $39 - 5 - 13 = 21$.

	5	
9	13	17
x	21	

Since the numbers in the bottom row add to 39, then the number in the bottom right square is $39 - 21 - x = 18 - x$.
Since the numbers in the bottom left to top right diagonal add to 39, then the number in the top right square is $39 - 13 - x + 26 - x$.
Since the numbers in the rightmost column add to 39, then $(26-x)+17+(18-x) = 39$ or $61 - 2x = 39$ or $2x = 22$, and so $x = 11$.
We can complete the magic square as follows:

19	5	15
9	13	17
11	21	7

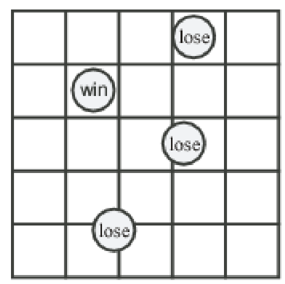
5. What is the largest positive integer n that satisfies $n^{200} < 3^{500}$?

{2010 Cayley #20}

Solution 1
Note that $n^{200} = (n^2)^{100}$ and $3^{500} = (3^5)^{100}$.
Since n is a positive integer, then $n^{200} < 3^{500}$ is equivalent to $n^2 < 3^5 = 243$.
Note that $15^2 = 225$, $16^2 = 256$ and if $n \geq 16$, then $n^2 \geq 256$.
Therefore, the largest possible value of n is 15.

Solution 2
Since n is a positive integer and 500=200(2.5), then $n^{200} < 3^{500}$ is equivalent to $n^{200} < (3^{2.5})^{200}$,
which is equivalent to $n < 3^{2.5} = 3^2 3^{0.5} = 9\sqrt{3}$.
Since $9\sqrt{3} \approx 15.59$ and n is an integer, the largest possible value of n is 15.

6. A coin that is 8 cm in diameter is tossed onto a 5 by 5 grid of squares each having side length 10 cm. A coin is in a winning position if no part of it touches or crosses a grid line, otherwise it is in a losing position. Given that the coin lands in a random position so that no part of it is off the grid, what is the probability that it is in a winning position?



{2010 Cayley #24}

Solution
Since the grid is a 5 by 5 grid of squares and each square has side length 10 cm, then the whole grid is 50 cm by 50 cm.
Since the diameter of the coin is 8 cm, then the radius of the coin is 4 cm.
We consider where the centre of the coin lands when the coin is tossed, since the location of the centre determines the position of the coin.
Since the coin lands so that no part of it is off of the grid, then the centre of the coin must land at least 4 cm (1 radius) away from each of the outer edges of the grid.
This means that the centre of the coin lands anywhere in the region extending from 4 cm from the left edge to 4 cm from the right edge (a width of 50-4-4=42 cm) and from 4 cm from the top edge to 4 cm to the bottom edge (a height of 50-4-4=42 cm).
Thus, the centre of the coin must land in a square that is 42 cm by 42 cm in order to land so that no part of the coin is off the grid.
Therefore, the total admissible area in which the centre can land is $42 \times 42 = 1764$ cm^2.
Consider one of the 25 squares. For the coin to lie completely inside the square, its centre must land at least 4 cm from each edge of the square.
As above, it must land in a region of width 10-4-4=2 cm and a height of 10-4-4=2 cm.
There are 25 possible such regions (one for each square) so the area in which the centre of the coin can land to create a winning position is $25 \times 2 \times 2 = 100$ cm^2.
Thus, the probability that the coin lands in a winning position is equal to the area of the region in which the centre lands giving a winning position, divided by the area of the region in which the coin amy land, or $\frac{100}{1764} = \frac{25}{441}$.

7. Daryl first writes the perfect squares as a sequence

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

After the number 1, he then alternates by making two terms negative followed by leaving two terms positive. Daryl's new sequence is

$$1, -4, -9, 16, 25, -36, -49, 64, 81, -100, \ldots$$

What is the sum of the first 2011 terms in this new sequence?

{2011 Gauss 8 #25}

Solution
The given sequence allows for many different patterns to be discovered depending on how terms in the sequence are grouped and then added.
One possibility is to add groups of four consecutive terms in the sequence.
That is, consider finding the sum of the sequence, S, in the manner shown below.

$$S = (1+(-4)+(-9)+16)+(25+(-36)+(-49)+64)+(81+(-100)+(-121)+144)+\ldots$$

The pattern that appears when grouping terms in this way is taht each consecutive group of 4 terms, beginning at the first term, adds to 4.
That is $1 + (-1) + (-9) + 16 = 4$, $25 + (-36) + (-49) + 64 = 4$, $81 + (-100) + (-121) + 144 = 4$, and so on.
For now, we will assume taht this pattern of four consecutive terms adding to 4 continues and wait to verify this at the end of the solution.
Since each consecutive group of four terms adds to 4, the first eight terms add to 8, the first twelve terms add to 12 and the first n terms add to n provided taht n is a multiple of 4.
Thus, the sum of the first 2012 terms is 2012, since 2012 is a multiple of 4.
Since we are required to find the sum of the first 2011 terms, we must subtract the value of the 2012^{th} term from our total of 2012.
We know that the n^{th} term in the sequence is either n^2 or it is $-n^2$.
Therefore, we must determine if the 2012^{th} term is positive or negative.
By the alternating pattern of the signs, the first and fourth terms in each of the consecutive groupings will be positive, while the second and third terms are negative.
Since the 2012^{th} is fourth in its group of four, its sign is positive.
Thus, the 2012^{th} term is 2012^2.
Therefore, the sum of the first 2011 terms is the sum of the first 2012 terms, which is 2012, less the 2012^{th} term which is 2012^2.
Thus, $S = 2012 = 2012^2 = 2012 - 4048144 = -4046132$

Verifying the Pattern
While we do not expect that students will verify this pattern in the context of a multiple choice contest, it is always important to verify patterns.
One way to verify that the sum of each group of four consecutive terms (beginning with the first term) adds to 4, is to use algebra.
If the first of four consecutive integers is n, then the next three integers in order are $n + 1$, $n + 2$, and $n + 3$.
Since the terms in our sequence are the squares of consecutive integers, we let n^2 represent the first term in the group of four.

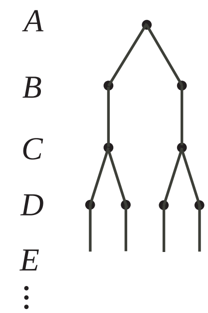
The square of the next integer larger than n is $(n+1)^2$, and thus the remaining two terms in the group are $(n+2)^2$ and $(n+3)^2$.
Since the first and fourth terms are positive, while the second and third terms are negative, the sum of the four terms is $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2$.
To simplify this expression, we must first understand how to simplify its individual parts such as $(n+1)^2$.
The expression $(n+1)^2$ means $(n+1) \times (n+1)$.
To simplify this product, we multiply the n in the first set of brackets by each term in the second set of brackets and do the same for the 1 appearing in the first set of brackets.
The operation between each product reamins as it appears in the expression, as an addition.
That is,

$$\begin{aligned} (n+1)^2 &= (n+1) \times (n+1) \\ &= n \times n + n \times 1 + 1 \times n + 1 \times 1 \\ &= n^2 + n + n + 1 \\ &= n^2 + 2n + 1 \end{aligned}$$

Applying this process again,
$(n+2)^2 = n^2 + 4n + 4$,
and
$(n+3)^2 = n^2 + 6n + 9$.
Therefore, the sum of each group of four consecutive terms (beginning with the first term) is,

$$\begin{aligned} n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 &= n^2 - (n^2 + 2n + 1) - (n^2 + 4n + 4) + (n^2 + 6n + 9) \\ &= n^2 - n^2 - 2n - 1 - n^2 - 4n - 4 + n^2 + 6n + 9 \\ &= 4 \end{aligned}$$

8. In the diagram, there are 26 levels, labelled $A, B, C, \ldots, Z$. There is one dot on level A. Each of levels $B, D, F, H, J, \ldots$, and Z contains twice as many dots as the level immediately above. Each of levels $C, E, G, I, K, \ldots$, and Y contains the same number of dots as the level immediately above. How many dots does level Z contain?



{2011 Pascal #21}

<u>Solution</u>
Since level C contains the same number of dots as level B and level D contains twice as many dots as level C, then level D contains twice as many dots as level B.
Similarly, level F contains twice as many dots as level D, level H contains twice as many dots as level F, and so on.
Put another way, the number of dots doubles from level B to level D, from level D to level F, from level F to level H, and so on.
Since there are 26 levels, then there are 24 levels after level B.
Thus the number of dots doubles $24 \div 2 = 12$ times from level B to level Z.
Therefore, the number of dots on level Z is $2 \times 2^{12} = 2^{13} = 8192$

9. An ordered list of four numbers is called a *quadruple*.
A quadruple (p, q, r, s) of integers with $p, q, r, s \geq 0$ is chosen at random such that

$$2p + q + r + s = 4$$

What is the probability that $p + q + r + s = 3$?

{2011 Pascal #23}

First, we count the number of quadruples (p, q, r, s) of non-negative integer solutions to the equation $2p + q + r + s = 4$. Then, we determine which of these satisfies $p + q + r + s = 3$. This will allow us to calculate the desired probability.
Since each of p, w, r, and s is a non-negative integers and $2p + q + r + s = 4$, then there are three possible values for $p : p = 2, p = 1$, and $p = 0$.
Note that, in each case, $q + r + s = 4 - 2p$.
Case 1: $p = 2$
Here, $q + r + s = 4 - 2(2) = 0$.
Since each of q, r, and s is non-negative, then $q = r = s = 0$, so $(p, q, r, s) = (2, 0, 0, 0)$.
There is 1 solution in this case.
Case 2: $p = 1$
Here, $q + r + s = 4 - 2(1) = 2$.
Since each of q, r, and s is non-negative, then the three numbers q, r, and s must be 0, 0, and 2 in some order or 1, 1, and 0 in some order.
There are three ways to arrange a list of three numbers, two of which are the same.
Therefore, the possible quadruples here are

$$(p, q, r, s) = (1, 2, 0, 0), (1, 0, 2, 0), (1, 0, 0, 2), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1)$$

There are 6 solutions in this case.

Case 3: $p = 0$
Here, $q + r + s = 4$.
We will look for non-negative integer solutions to this equation with $q \geq r \geq s$. Once we have found these solutions, all solutions can be found by re-arranging these initial solutions.
If $q = 4$, then $r + s = 0$, so $r = s = 0$.
If $q = 3$, then $r + s = 1$, so $r = 1$ and $s = 0$.
If $q = 2$, then $r + s = 2$, so $r = 2$ and $s = 0$, or $r = s = 1$.
The value of q cannot be 1 or 0, because if it was, then $r + s$ would be at least 3 and so r or s would be at least 2. (We are assuming that $r \leq q$ so this cannot be the case.)
Therefore, the solutions to $q + r + s = 4$ must be the three numbers 4, 0, and 0 in some order, 3, 1, and 0 in some order, 2, 2, and 0 in some order, or 2, 1, and 1 in some order.
The solution $(p, q, r, s) = (0, 4, 0, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 3, 1, 0)$ has 6 arrangements.
The solution $(p, q, r, s) = (0, 2, 2, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 2, 1, 1)$ has 3 arrangements.
There are 15 solutions in this case.

Overall, there are $1 + 6 + 15 = 22$ solutions to $2p + q + r + s = 4$.
We can go through each of these quadruples to check which satisfy $p+q+r+s = 3$.
The quadruples that satisfy this equation are exactly those from Case 2.
Therefore, of the 22 solutions to $2p+q+r+s = 4$, there are 6 that satisfy $p+q+r+s = 3$, so the desired probability is $\frac{6}{22} = \frac{3}{11}$.

10. Let n be the largest integer for which $14n$ has exactly 100 digits. Counting from right to left, what is the 68th digit of n?

{2011 Pascal #24}

Solution
The largest integer with exactly 100 digits is the integer that consists of 100 copies of the digit 9.
This integer is equal to $10^{100} = 1$.
Therefore, we want to determine the alrgest integer n for which $14n < 10^{100} - 1$.
This is the same as trying to determine the alrgest integer n for which $14n < 10^{100}$, since $14n$ is an integer.
We want to find the alrgest integer n for which $n < \frac{10^{100}}{14} = \frac{10}{14} \times 10^{99} = \frac{5}{7} \times 10^{99}$.
This is equivalent to calculating the number $\frac{5}{7} \times 10^{99}$ and rounding down to the nearest integer.
Put another way, this is the same as calculating $\frac{5}{7} \times 10^{99}$ and truncating the number at the decimal point.
The decimal expansion of $\frac{5}{7}$ is $0.\overline{714285}$.
Therefore, the integer that we are looking for is the integer obtained by multiplying $0.\overline{714285}$ by 10^{99} and truncating at the decimal point.
In other words, we are looking for the integer obtained by shifting the decimal point in $0.\overline{714285}$ by 99 places to the right, and then ignoring everything after the new decimal point.
Since the digits in the decimal expansion repeat with period 6, then the integer consists of 16 copies of the digits 714285 followed by 714. (This has $16 \times 6 + 3 = 99$

digits).
We must determine the digits that is the 68^{th} digit from the right.
If we start listing groups from the right, we first ahve 714 (3 digits) followed by 11 copies of 714285 (66 more digits). This is 69 digits in total.
Therefore, the 7 that we have arrived at is the 69^{th} digit from the right.
Moving one digit back towards the right tells us that the 68^{th} digit from the right is 1.

11. (a) Determine the average of the integers $71, 72, 73, 74, 75$.

Solution
The average of any sets of integers is equal to the sum of the integers divided by the number of integers.
Thus, the average of the integers is $\frac{71+72+73+74+75}{5} = \frac{365}{5} = 73$.

(b) Suppose that $n, n+1, n+2, n+3, n+4$ are five consecutive integers.

(i) Determine a simplified expression for the sum of these five consecutive integers.

Solution
Simplifying, $n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 10$.

(ii) If the average of these five consecutive integers is an odd integer, explain why n must be an odd integer.

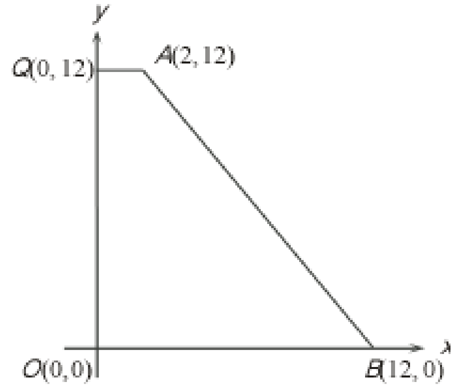
Solution
Since the sum of the 5 consecutive integers is $\frac{5n+10}{5} = n + 2$.
If n is an even integer, then $n+2$ is an even integer.
If n is an odd integer, then $n+2$ is an odd integer.
Thus for the average $n+2$ to be odd, the integer n must be odd.

(c) Six consecutive integers can be represented by $n, n+1, n+2, n+3, n+4, n+5$, where n is an integer. Explain why the average of six consecutive integers is never an integer.

Solution
Simplifying, $n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) = 6n + 15$.
Since the sum of the 6 consecutive integers is $6n + 15$, the average of these integers is $\frac{6n+15}{6} = n + \frac{15}{6} = n + \frac{5}{2}$.
For every integer n, $n + \frac{5}{2}$ is never an integer.
Therefore, the average of six consecutive integers is never an integer.

{2010 Fryer #2}

12. (a) Quadrilateral $QABO$ is constructed as shown. Determine the area of $QABO$.



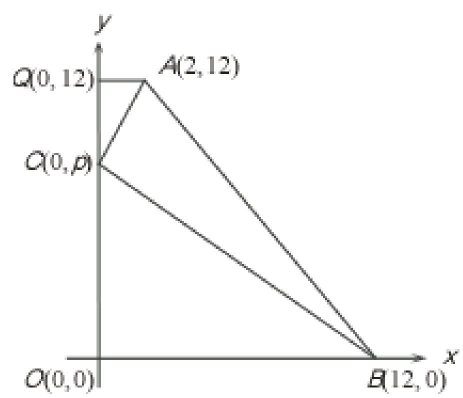
Solution 1

If point T is placed at (2, 0), then T is on OB and AT is perpendicular to OB.
Since QO is perpendicular to OB, then QO is parallel to AT.
Both QA and OT are horizontal, so then QA is parallel to OT.
Therefore, $QATO$ is a rectangle.
The area of rectangle $QATO$ is $QA \times QO$ or $(2-0) \times (12-0) = 24$
Since AT is perpendicular to TB, we can treat AT as the height of $\triangle ATB$ and TB as the base.
The area of $\triangle ATB$ is $\frac{1}{2} \times TB \times AT$ or $\frac{1}{2} \times (12-2) \times (12-0) = \frac{1}{2} \times 10 \times 12 = 60$.
The area of $QABO$ is the sum of the areas of rectangle $QATO$ and $\triangle ATB$, or 24+60=84.

Solution 2

Both QA and OB are horizontal, so then QA is parallel to OB.
Thus, $QABO$ is a trapezoid.
Since QO is perpendicular to OB, we can treat QO as the height of the trapeziod.
Then $QABO$ has area $\frac{1}{2} \times QO \times (QA + OB) = \frac{1}{2} \times 12 \times 14 = 84$.

(b) Point $C(0, p)$ lies on the y-axis between $Q(0, 12)$ and $O(0, 0)$ as shown. Determine an expression for the area of $\triangle COB$ in terms of p.



Solution

Since CO is perpendicular to OB, we can treat CO as the height of $\triangle COB$ and OB as the base. The area of $\triangle COB$ is $\frac{1}{2} \times OB \times CO$ or $\frac{1}{2} \times (12-0) \times (p-0) = \frac{1}{2} \times 12 \times p = 6p$.

(c) Determine an expression for the area of $\triangle QCA$ in terms of p.

Solution

Since QA is perpendicular to QC, we can treat QC as the height of $\triangle QCA$ and QA as the base. The area of $\triangle QCA$ is $\frac{1}{2} \times QA \times QC$ or $\frac{1}{2} \times (2-0) \times (12-p) = \frac{1}{2} \times 2 \times (12-p) = 12 - p$.

(d) If the area of $\triangle ABC$ is 27, determine the value of p.

Solution

The area of $\triangle ABC$ can be found by subtracting the area of $\triangle COB$ and the area of $\triangle QCA$ from the area of quadrilateral $QABO$.
From parts (a), (b), and (c), the are of $\triangle ABC$ is thus $84-6p-(12-p) = 72-5p$.
Since the area of $\triangle ABC$ is 27, then $72 - 5p = 27$ so $p = 9$.

{2010 Galois #2}

13. If m is a positive integer, the symbol $m!$ is used to represent the product of the integers from 1 to m. That is, $m! = m(m-1)(m-2)\ldots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$. Some positive integers can be written in the form

$$n = a(1!) + b(2!) + c(3!) + d(4!) + e(5!).$$

In addition, each of the following conditions is satisfied:

- $a, b, c, d,$ and e are integers
- $0 \le a \le 1$
- $0 \le b \le 2$
- $0 \le c \le 3$
- $0 \le d \le 4$
- $0 \le e \le 5$.

(a) Determine the largest positive value of N that can be written in this form.

Solution

The largest positive integer N that can be written in this form is obtained by maximizing the values of the integers $a, b, c, d,$ and e. Thus, $a = 1, b = 2, c = 3, d = 4,$ and $e = 5$, which gives $N = 1(1!) + 2(2!) + 3(3!) = 4(4!) + 5(5!) = 1 + 2(2) + 3(6) + 4(24) + 5(120) = 719$.

(b) Write $n = 653$ in this form.

Solution

For any two positive integers n and m, it is always possible to write a division statement of the form,

$$n = qm + r$$

where the quotient q and remainder r are non-negative integers and $0 \le r < m$. The following table shows some examples of this:

n	m	q	r	$n = qm + r$
20	6	3	2	20=3(6)+2
12	13	0	12	12=0(13)+12
9	7	1	2	9=1(7)+2
36	9	4	0	36=4(9)+0

Notice that in each of the 4 examples, the inequality $o \le r < m$ has been satisfied.

We can always satisfy this inequality by beginning with n and then subtracting multiples of m from it until we get a number in the range 0 to $m-1$. We let r be this number, or $r = n - qm$, so that $n = qm + r$.
Further, this process is repeatable. For example, beginning with $n = 653$ and $m = 5! = 120$, we get 653=5(120)+53. We can now repeat the process using remainder $r = 53$ as our next n, and 4!=24 as our next m. This process is shown in the table below with each new remainder becoming our next n and m taking the successive values of 5!, 4!, 3!, 2!, and 1!.

n	m	q	r	$n = qm + r$
653	120	5	53	653=5(120)+53
53	24	2	5	53=2(24)+5
5	6	0	5	5=0(6)+5
1	1	1	0	1=1(1)+0

From the 5$^{\text{th}}$ column of the table above,

$$\begin{aligned} 653 &= 5(120) + 53 \\ &= 5(120) + 2(24) + 5 \\ &= 5(120) + 2(24) + 0(6) + 5 \\ &= 5(120) + 2(24) + 0(6) + 2(2) + 1 \\ &= 5(120) + 2(24) + 0(6) + 2(2) + 1(1) + 0 \\ &= 5(5!) + 2(4!) + 0(3!) + 2(2!) + 1(1!) \end{aligned}$$

Thus, $n = 653$ is written in the required form with $a = 1, b = 2, c = 0, d = 2$ and $e = 5$.

(c) Prove that all integers n, where $0 \leq n \leq N$, can be written in this form.

Solution
Following the process used in (b) above, we obtain the more general result shown here.

n	m	q	r	$n = qm + r$	restriction on r
n	120	e	r_1	$n = e(120) + r_1$	$0 \leq r_1 < 120$
r_1	24	d	r_2	$r_1 = d(24) + r_2$	$0 \leq r_2 < 24$
r_2	6	c	r_3	$r_2 = c(6) + r_3$	$0 \leq r_3 < 6$
r_3	2	b	r_4	$r_3 = b(2) + r_4$	$0 \leq r_4 < 2$
r_4	1	a	r_5	$r_4 = a(1) + r_5$	$0 \leq r_5 < 1$

From the 5$^{\text{th}}$ column of this table,

$$\begin{aligned} n &= e(120) + r_1 \\ n &= e(120) + d(24) + r_2 \\ n &= e(120) + d(24) + c(6) + r_3 \\ n &= e(120) + d(24) + c(6) + b(2) + r_4 \\ n &= e(120) + d(24) + c(6) + b(2) + a(1) + r_5 \\ n &= e(5!) + d(4!) + c(3!) + b(2!) + a(1!) \end{aligned}$$

We must justify that the integers a, b, c, d, and e satisfy their required inequality. From part (b), each of these quotients is a non-negative integer. Therefore, it remains to show that $a \leq 1, b \leq 2, c \leq 3, d \leq 4$, and $e \leq 5$.
From part (a), $N = 719$, therefore $0 \leq n < 720$.
From the table above, we have $n = e(120) + r_1$. Therefore $e(120) + r_1 < 720$ or $e(120) < 720$ (since $r_1 \geq 0$), and so e¡6.
Thus $e \leq 5$, as required.
Also from the table above, $r_1 < 120$, so $d(24) + r_2 < 120$ or $d(24) < 120$ (since $r_2 \geq 0$), and therefore $d < 5$.
Thus, $d \leq 24$, as required.
Also, $r_2 < 24$, so $c(6)+r_3 < 24$ or $c(6) < 24$ (since $r_3 \geq 0$), and therefore $c < 4$.
Thus, $c \leq 3$, as required.
Continuing, $r_3 < 6$, so $b(2) + r_4 < 6$ or $b(2) < 6$ (since $r_4 \geq 0$), and therefore $b < 3$.
Thus, $b \leq 2$, as required.
Finally, $r_4 < 2$, so $a(1) + r_5 < 2$ or $a(1) < 2$ (since $r_5 = 0$), and therefore a¡2.
Thus, $a \leq 1$, as required.
Therefore, all integers n, with $0 \leq n \leq N$,can be written in the required form.

(d) Determine the sum of all integers n that can be written in this form with $c = 0$.

Solution

Since $c = 0$, we are required to find the sum of all integers n of the form $n = a + 2b + 24d + 120e$, with the stated restrictions on the integers a, b, d, and e.
Since $n = a+2b+24d+120e = (a+2b)+24(d+5e)$, let $n_1 = a+2b$ and $n_2 = d+5e$ so that $n = n_1 + 24n_2$. First, consider all possible values of n_1.
Since $0 \leq a \leq 1$ and $0 \leq b \leq 2$ and $n_1 = a + 2b$, we have that n_1 can equal any of the numbers in the set {0, 1, 2, 3, 4, 5}. Each of these comes from exactly one pair (a, b).
Next, find all possible values for $n_2 = d + 5e$. Since $0 \leq d \leq 4$ and $0 \leq e \leq 5$, we have that $d + 5e$ can equal any of the numbers in the set {0, 1, 2, 3, 4, 5, 6, 7, . . . , 29}.
Each of these comes from exactly one pair (d, e).
Therefore, $24n_2$ can equal any of the numbers in the set
$\{24 \times 0, 24 \times 1, 24 \times 2, \ldots, 24 \times 29\} = \{0, 24, 48, \ldots, 696\}$, the multiples of 24 from 0 to 696.
Adding each of these possible values of $24n_2$ in turn to each of the 6 possible values of n_1, we get the set of all possible $n = n_1 + 24n_2$:

$$\{0, 1, 2, 3, 4, 5, 24, 25, 26, 27, 28, 29, 48, 49, 50, 51, 52, 53, \ldots, 696, 697, 698, 699, 700, 701\}$$

Because each of the 6 possible values of n_1 comes from exactly one pair (a, b) and each of the 30 possible values of n_2 comes from exactly one pair (d, e), then each of these integers above occurs exactly once as a, b, d, and e move through their possible values. It remains to find the sum of the possible values for n:

$$\begin{aligned}
& 0+1+2+3+4+5+24+25+26+27+28+29+48+48+\cdots+699+700+701 \\
= \; & 0+1+2+3+4+5+(24+0)+(24+1)+(24+2)+(24+3)+(24+4)+(24+5) \\
& +(48+0)+(48+1)+\cdots+(696+3)+(696+4)+(696+5) \\
= \; & (0+1+2+3+4+5)+24\times 6+(0+1+2+3+4+5)+48\times 6+(0+1+2+3+4+5) \\
& +\cdots+696\times 6+(0+1+2+3+4+5) \\
= \; & 30(0+1+2+3+4+5)+24\times 6+48\times 6+\cdots+696\times 6 \\
= \; & 30(15)+24(6)[1+2+3+\cdots+29] \\
= \; & 30(15)+24(6)\left[\frac{29\times 30}{2}\right] \\
= \; & 63090
\end{aligned}$$

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