UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 9 & 10 Math Circles
November 23, 2011
Pascal Contest Questions - 1988

(1) What is the value of $\frac{(0.2)^2}{2}$?

Solution

$$\begin{aligned} \frac{(0.2)^2}{2} &= \frac{0.04}{2} \\ &= 0.02 \end{aligned}$$

(2) A family has 6 children whose ages total 36. In three years, what will the ages of the children total?

Solution

In three years, each of the 6 children will be 3 years older. This means the ages of the children will total $36 + (6 \cdot 3) = 54$

(3) If $x = -2$, what is the least number in the set $\{2x, -4x, x^2, \frac{4}{x}, \frac{0}{x}\}$?

Solution

$$\begin{aligned} 2x &= 2(-2) \\ &= -4 \end{aligned}$$

$$\begin{aligned} -4x &= -4(-2) \\ &= 8 \end{aligned}$$

$$\begin{aligned} x^2 &= (-2)^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \frac{4}{x} &= \frac{4}{-2} \\ &= -2 \end{aligned}$$

$$\frac{0}{x} = 0$$

Therefore, the least number in the set is $2x$.

(4) If the 24^{th} day of June was on a Thursday, then what day was the first day of June, in the same year, on?

Solution
There are $24 - 1 = 23$ days between these two days. Since there is 7 days in a week, this means there is 3 weeks and 2 $(23 - 3 \cdot 7 = 2)$ days between these two dates. Since each of the 3 weeks will bring us back to a Thursday, we only need to worry about the remaining 2 days. 2 days before a Thursday is a Tuesday.
Therefore the first day of June, in the same year, will be on a Tuesday.

(5) Two angles are supplementary with one angle 50° greater than the other. What is the smaller angle, in degrees?

Solution
Recall that two angles are supplementary if and only if the sum of the two angles is 180°.

Let x be the smaller angle.

$$\begin{aligned}(x + 50) + x &= 180 \\ 2x &= 130 \\ x &= 65\end{aligned}$$

Therefore the smaller angle is 65°.

(6) It requires 3 litres of paint to cover a floor. If each dimension of the floor is doubled, then what is the number of litres of paint required to cover the floor with the same thickness of paint?

Solution
Let l be the length of the floor and w be the width of the floor.
It takes 3 litres to cover an area of lw.
If each dimension is doubled, the new area to cover is $2l \cdot 2w = 4lw$.
This means it will take 4 times as much paint to cover the floor.
$4(3) = 12$
Therefore it will take 12 litres of paint to cover the floor.

(7) A classroom contained an equal number of boys and girls. Eight girls left to play basketball, leaving twice as many boys as girls in the classroom. What was the original number of students that were present?

Solution Let x be the orginal number of girls (which is also the original number of boys) in the classroom.
This means we want to find $x + x = 2x$.

$$\begin{aligned}2(x - 8) &= x \\ 2x - 16 &= x \\ x &= 16 \\ \\ 2x &= 32\end{aligned}$$

Therefore there were orginally 32 students present.

(8) If $5x - 3 = 5$, then what does $10x - 10$ equal?

Solution

$$\begin{aligned}
5x - 3 &= 5 \\
5x &= 8 \\
x &= \frac{8}{5} \\
10x - 10 &= 10\left(\frac{8}{5}\right) - 10 \\
&= 16 - 10 \\
&= 6
\end{aligned}$$

(9) If n is a positive integer, then what is the number of values of n which satisfy $0 < n^2 < 1\ 000$?

Solution

$0 < n^2$ for any value of n, so we only have the right side of the inequality to take care of.

$$\begin{aligned}
n^2 &< 1\ 000 \\
\sqrt{n^2} &< \sqrt{1\ 000} \\
n &< 31.62
\end{aligned}$$

Since n is an integer, $0 < n \leq 31$.

Therefore there are 31 integers that satisfy this inequality.

(10) Tara had an average of 56 marks on her first 7 exams. To obtain an average of 60 marks on 8 exams, what must be mark on the eighth exam be?

Solution

Let m_n be the mark of the n^{th} test.

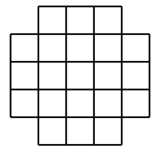
$$\begin{aligned}
\frac{m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7}{7} &= 56 \\
m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 &= 392
\end{aligned}$$

So the total mark for the first 7 tests was 392. To get the new average, we must have:

$$\begin{aligned}
\frac{m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8}{8} &= 60 \\
\frac{392 + m_8}{8} &= 60 \\
392 + m_8 &= 480 \\
m_8 &= 88
\end{aligned}$$

Therefore Tara must get 88% to obtain an average of 60 marks.

(11) What is the total number of squares of all sizes that appear in the diagram?



Solution

There are 21 small 1×1 squares.
Counting the 2×2 squares, we can see that there are 12 2×2 squares.
There are 5 3×3 squares.

Adding these up we get: $21 + 12 + 5 = 38$ squares.

(12) If $b = 4d$, $c = 2d$, and $b + c + d = 42$, then what does b equal?

Solution

$$\begin{aligned} b &= 4d && (1) \\ c &= 2d && (2) \\ b + c + d &= 42 && (3) \end{aligned}$$

Substituting (1) and (2) into (3):

$$\begin{aligned} 4d + 2d + d &= 42 \\ 7d &= 42 \\ d &= \frac{42}{7} \\ &= 6 \end{aligned}$$

Substituting this back into (1):

$$\begin{aligned} b &= 4(6) \\ &= 24 \end{aligned}$$

(13) A bag contains 80 jellybeans, 20 of which are red, 20 are black, 20 are green, and 20 are yellow. What is the least number that a blindfolded person must eat to be certain of having eaten at least one of every colour?

Solution

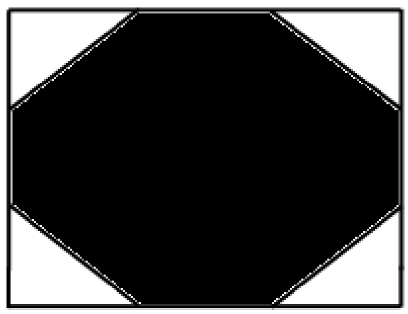
The maximum number of jellybeans that a person could eat where they have only ate 3 colours is $20 \cdot 3 = 60$. This means that the person has eaten every jellybean of the first 3 colours. If they eat one more, they will have eaten the last colour.
Therefore, the least number that a blindfolded person must eat to be certain of having eaten at least one of each colour is $60 + 1 = 61$.

(14) If the reciprocal of $\left(\frac{1}{x}-1\right)$ is -2, then what does x equal?

Solution

$$\begin{aligned}
\frac{1}{\left(\frac{1}{x}-1\right)} &= -2 \\
1 &= -2\left(\frac{1}{x}-1\right) \\
1 &= \frac{-2}{x}+2 \\
x &= -2+2x \\
x &= 2
\end{aligned}$$

(15) The sides of a 12×9 rectangle are trisected. the points are joined as shown in the diagram. What is the perimeter of the shaded octagon?



Solution

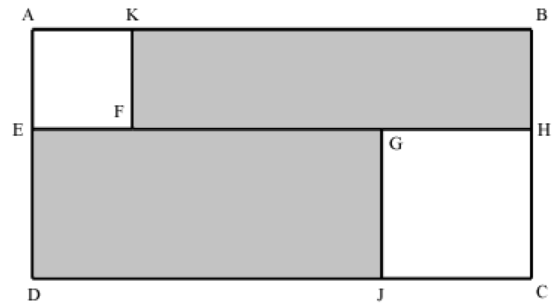
The top and bottom edge of the octagon has length $\frac{12}{3}=4$.
The two side edges of the octagon has length $\frac{9}{3}=3$.
To find the length of the four remaining edges, we can use Pythagorean Theorem.

$$\begin{aligned}
4^2+3^2 &= c^2 \\
16+9 &= c^2 \\
25 &= c^2 \\
c &= 5
\end{aligned}$$

$$\begin{aligned}
4\cdot 5+2\cdot 4+2\cdot 3 &= 20+8+6 \\
&= 34
\end{aligned}$$

Therefore the perimeter of the shaded octagon is 34 units.

(16) A rectangle $ABCD$ has a square $AEFK$ of area 4, and a square $GHCJ$ of area 9 removed. If $EFGH$ is a straight line segment of $FG = 5$, then what is the total area of the two shaded rectangles?



Solution

Since the area $AEFK$ is 4, and $AEFK$ is a square, this means that $AE = AK = \sqrt{4} = 2$.
Similarily, since the area of $GHCJ$ is 9, and $GHCJ$ is a square, this means that $JC = CH = \sqrt{9} = 3$.
The length of the rectangle is $AE + CH = 2 + 3 = 5$.
The width of the rectangle is $AK + FG + GH = 2 + 5 + 3 = 10$.
This means that the area of the entire rectangle is $5(10) = 50$.

Since we only want the area of the shaded region, we must find the difference between the entire rectangle and the two squares.
i.e. 50-4-9=37

Therefore, the area of the two shaded rectangles is 37 units squares.

(17) If 630 is expressed as the product of three positive integers whose sum is 38, then what is the largest of these integers?

Solution

Using prime factorization we get that $630 = 2 \times 3^2 \times 5 \times 7$.
We can now guess and check to find three numbers that satisfy the given restraints.
Three numbers that satify the above are 15, 21 and 2.
The largest of three three integers is 21.

(18) In a sequence of six numbers, the first number is 4 and the last number is 47. Each number after the second equals the sum of the previous two numbers. If S is the sum of the six numbers in this sequence, what is S?

Solution

Our sequence looks like this:

4	x	4+x	4+2x	8+3x	47

Note: the last term is 47 but it is also

$$(4+2x)+(8+3x) = 12+5x$$

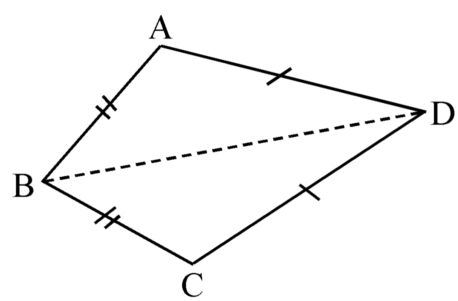
$$\begin{aligned} 12+5x &= 47 \\ 5x &= 35 \\ x &= 7 \end{aligned}$$

So the sum of the 6 terms is $4+7+4+7+4+2(7)+8+3(7)+47 = 116$

(19) In quadrilateral $ABCD$, $AB = BC$ and $CD = DA$. If $\angle ABC = 50°$ and $\angle BAD = 120°$ then what is $\angle ADC$, in degrees?

Solution

Using the given information, we get a quadrilateral that resembles



The triangle above the dotted line and the triangle below the dotted line are the same as two of the sides are the same length.
Looking at the top triangle ($\triangle ABD$), we know that all the angles in a triangle must sum to $180°$.
This means that $120 + \frac{50}{2} + \frac{x}{2} = 180$, where x is $\angle ADC$.

$$\begin{aligned} 120 + \frac{50}{2} + \frac{x}{2} &= 180 \\ 120 + 25 + \frac{x}{2} &= 180 \\ 145 + \frac{x}{2} &= 180 \\ \frac{x}{2} &= 35 \\ x &= 70 \end{aligned}$$

Therefore $\angle ADC = 70°$.

(20) A vertical line intersects the symbol S at three distinct points. The line divides the S into four parts as numbered in the diagram. If the S is intersected in this way by nine parallel lines, each of which intersects the S at three points, then what is the number of parts?

Solution

It can be seen that when the S is intersected this way with 2 parallel lines, there are 7 parts. Similarily, when the S is intersected this way with 3 parallel lines, there are 10 parts.

With each line that is added, 3 more parts are added (as the middle section has three parts). Therefore, if the S is intersected this way by nine parallel lines, then the number of parts is $4 + (8 \times 3) = 4 + 24 = 28$.

(21) A number of 1 cm cubes are glued together to construct the sides and bottom of an open box. If the outside dimensions of the box are 10 cm $\times$ 10 cm $\times$ 10 cm and the sides and bottom are 1 cm thick, then what is the number of cubes used?

Solution

The front and back sides of the box will both be made up of $10 \times 10 = 100$ cubes.

The left and right sides of the box each share two columns of cubes with the front and back sides, so they will both be made up of $8 \times 10 = 80$ cubes.

Finally, the bottom face of the box shares its border with the four sides of the box, and so will be made up of $8 \times 8 = 64$ additional cubes.

Thus, the number of cubes used to make the box is $2 \times 100 + 2 \times 80 + 64 = 424$.

(22) If 5^{101} is expressed as an integer, what are the last three digits of the integer?

Solution

Listing the first few results of 5^n we get:

$$\begin{aligned}
5^1 &= 5\\
5^2 &= 25\\
5^3 &= 125\\
5^4 &= 625\\
5^5 &= 3\ 125\\
5^6 &= 15\ 625\\
5^7 &= 78\ 125
\end{aligned}$$

and so on ...

Notice that as we get into larger exponents, the last three digits are the same when the exponent is even and when the exponent is odd.

Since 101 is odd, the last three digits are 125.

(23) The digits $1, 2, 3, 4, 5$, and 6 are each used once to compose a six digit number $abcdef$, such that the three digit number abc is divisible by 4, bcd is divisible by 5, cde is divisible by 3, and def is divisble by 11. What is the digit a?

Solution

Since bcd is divisible by 5, this means that d must either be 0 or 5. 0 is not an option, so $d = 5$.

It is important to know the divisibility rules to go on from here:

- A number n is divisible by 3 if the sum of the digits is divisible by 3
- A number n is divisible by 4 if the last two digits are divisible by 4
- A number n is divisible by 11 if the difference between the sum of the even numbered digits and the sum of the odd numbered digits is divisible by 11

Looking at abc, we know that the last two digits (bc) must be divisible by 4. This means that bc is also an even number, which means c can be $2, 4$, or 6.

If $c = 2$, then the only options for b are 1 or 3
but by our divisibility law for 3, $c + e + 5$ must be divisible by 3.
If $c = 2$, there are no options for e that will make this statement true.

If $c = 4$, then the only options for b are 2 or 6
but by our divisibility law for 3, $c + e + 5$ must be divisible by 3.
If $c = 4$, e must be either 3 or 6

If $c = 6$, then the only options for b are 1 or 3
but by our divisibility law for 3, $c + e + 5$ must be divisible by 3.
If $c = 6$, e must be either 1 or 4

Now, using the divisibility law for 11, we know that $(5 + f) - e$ must be divisible by 11.
If $e = 3 \Rightarrow (5 + f) - 3 = 2 + f$ must be divisible by 11. There is no number that is valid that will make this true, so $e \neq 3$.

If $e = 1 \Rightarrow (5 + f) - 1 = 4 + f$ must be divisible by 11. There is no number that is valid that will make this true, so $e \neq 1$.

If $e = 4 \Rightarrow (5 + f) - 4 = 1 + f$ must be divisible by 11. There is no number that is valid that will make this true, so $e \neq 4$.

This means that $e = 6$.

Since the only two options for c were 4 and 6, and c can no longer be 6, $c = 4$.

Now, since $c = 4$, the only two options we had for this option were $b = 2$ or $b = 6$. Since b cannot be 6, $b = 2$.

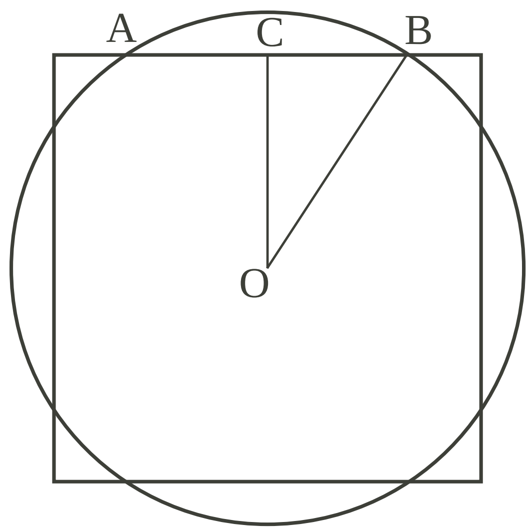
Coming back to the divisibility by 11 rule, since $e = 6$, we have that $(5 + f) - 6 = f - 1$ must be divisible by 11. The only options left for f are 1 and 3. This means that $f = 1$.

And finally, $a = 1$, as it is the only valid number left.

(24) A circle and a square, both centred at O, have equal areas. The circle intersects one side of the square at points A and B. If the radius of the circle is 1, then what is the length of the line segment AB?

Solution

We can draw a diagram of the circle and square and define a new point C to be the midpoint of AB, as shown.



Since the circle has a radius of 1, it has an area of π. This means that the square also has an area of π, so it must have side lengths of $\sqrt{\pi}$.

We know that the length of OB is 1, as it is a radius of the circle. We also know that the length of OC is $\frac{\sqrt{\pi}}{2}$, as it is half of the side length of the square.

Using the Pythagorean theorem, we find the length of CB to be $\sqrt{1^2 - (\frac{\sqrt{\pi}}{2})^2} = \sqrt{1 - \frac{\pi}{4}}$

This is half the length of AB, therefore the length of AB is equal to $2\sqrt{1 - \frac{\pi}{4}}$.

(25) If $P = 64 \times 63 \times 62 \times \cdots \times 3 \times 2 \times 1$, then what is the largest positive integer n, such that P is divisible by 12^n?

Solution

The prime factorization of 12 is $2^2 \times 3$, so it will be useful to know how many copies of 2 and 3 exist in P.

We begin by looking at multiples of 2: P has 32 of them, namely 2, 4, 6, $\ldots$, 62, 64. Each of these numbers contains at least one copy of 2.

The multiples of 4 however, contain two copies of 2, and P has 16 multiples of 4, and so we add 16 more copies of 2 to the 32 we already counted.

Multiples of 8 will each contain a third copy of 2, and as there are 8 multiples of 8 in P we had 8 more copies of 2 to our total.

Similarly, we add four more copies of 2 for the four multiples of 16 in P, then 2 more for the multiples of 32, and finally one more for the multiple of 64.

This gives us a total of $32 + 16 + 8 + 4 + 2 + 1 = 63$ copies of 2 present in P.

We can take a similar approach for the copies of 3. P contains 21 multiples of 3, 7 multiples of 9, and 2 multiples of 27. These add up to a total to 30 copies of 3 present in P.

The 63 copies of 2 tell us that we can have at most 32 copies of 12 present in P (since each copy of 12 requires two 2's). But since each copy of 12 also requires one 3, P must contain 30 copies of 12.

Therefore, $n = 30$.