Intermediate Math Circles
Problem Set Solutions
October 12, 2011

1. Alf is in Apsley, Ontario and Barb is in Buffalo, New York, which is 360 km away from Apsley. At 9:00 a.m. Alf starts driving towards Buffalo, at 80 km/h and Barb starts driving towards Apsley at 40 km/h. At what time do they pass each other, and how far away from Apsley are they at this point? (You may assume no delays in the border crossing.)

   Let t represent the time, in hours, until they pass each other.

   Since Alf drives 80 km/h for t hours, he drives $80t$ km. Since Barb drives 40 km/h for t hours, she drives $40t$ km. When they meet they will have driven the entire 360 km between them.

   $\therefore$ $80t + 40t = 360$ and $120t = 360$. Then $t = 3$ follows.

   Since $t = 3$, $80t = 80(3) = 240$. This is how far Alf has driven and is how far they are from Apsley.

   Alf and Barb pass each other after 3 hours, 240 km from Apsley.

2. Wendy leaves Whitby at 5:00 p.m. and is supposed to meet her friend Wanda in Waterloo, 150 km away, at 7:00 p.m. She gets stuck in traffic for the first half hour and only manages an average speed of 30 km/h during that time. What must her average speed be for the rest of the trip in order to make it to Waterloo on time?

   Driving for $\frac{1}{2}$ h at 30 km/h, Wendy drives $0.5 \times 30 = 15$ km. This leaves her $150 - 15 = 135$ km to drive in $2 - 0.5 = 1.5$ hours.

   Using $speed = distance \div time$, her average speed must be $135 \text{ km} \div 1.5\text{h} = 90\ \frac{\text{km}}{h}$ in order to arrive on time in Waterloo.

3. Janice's backyard pool springs a leak and begins losing water at a rate of 2 litres per minute. After an hour Janice notices the water level is lower than usual and begins filling the pool with a hose that outputs water at a rate of 10 litres per minute. If the pool continues to leak while she is refilling it, how long will it take for the pool to be full again?

   Since the pool loses $2\ \frac{\text{L}}{\text{min}}$, in one hour (60 minutes) it will lose $2\ \frac{\text{L}}{\text{min}} \times 60 \text{ min} = 120$ L of water.

   Janice then starts pumping water back in to the pool at a rate of $10\ \frac{\text{L}}{\text{min}}$ while the pool is still losing $2\ \frac{\text{L}}{\text{min}}$. The net gain of water into the pool is $10 - 2 = 8\ \frac{\text{L}}{\text{min}}$. To fill the pool to its original level 120 L must be replaced at a rate of $8\ \frac{\text{L}}{\text{min}}$. $\therefore$ it will take $120\text{L} \div 8\ \frac{\text{L}}{\text{min}} = 15$ min.

   Janice will refill the pool in 15 minutes (and she will still have to plug the leak).

4. Emma and Julie start at the same point on a 500 m circular track and run in opposite directions at constant but different speeds. They pass each other, running in opposite directions, after 54 seconds. If Emma, who is the faster runner, completes a lap of the track in 90 seconds, how long does it take Julie to complete a lap?

   Let Emma's speed be $(s_1)\ \frac{\text{m}}{\text{s}}$ and Julie's speed be $(s_2)\ \frac{\text{m}}{\text{s}}$ such that $s_1 > s_2$.

   In 54 s they run the full length of the track between them. So $54(s_1 + s_2) = 500$ and $s_1 + s_2 = \frac{500}{54} = \frac{250}{27}\ \frac{\text{m}}{\text{s}}$ follows. (1)

   But Emma runs 500 m in 90 s. Therefore $s_1 = \frac{500\ \text{m}}{90\ \text{s}} = \frac{50}{9}\ \frac{\text{m}}{\text{s}}$.

   Since $s_1 = \frac{50}{9}$ and from (1), $s_1 + s_2 = \frac{250}{27}$, $s_2 = \frac{250}{27} - \frac{50}{9} = \frac{250}{27} - \frac{150}{27} = \frac{100}{27}\ \frac{\text{m}}{\text{s}}$.

   Using *time* = *distance* ÷ *speed*, it takes $500\ \text{m} \div \frac{100}{27}\ \frac{\text{m}}{\text{s}} = 500\ \text{m} \times \frac{27}{100}\ \frac{\text{s}}{\text{m}} = 135$ s for Julie to run a complete lap.

5. Two candles of equal length are lit at noon. One candle takes 9 hours to completely burn while the other takes 6 hours to completely burn. At what time will the slower burning candle be exactly twice as long as the faster burning one?

   Let L represent the original length of each candle.

   Let t represent the time, in hours, until the slower burning candle is twice the height of the faster burning candle.

   The faster candle burns completely in 6 hours so in 1 hour, $\frac{1}{6}$ of the candle burns. In t hours, $\frac{1}{6}tL$ burns leaving $L - \frac{1}{6}tL$ of the candle. Similarly, the slower candle burns completely in 9 hours so in 1 hour, $\frac{1}{9}$ of the candle burns. In t hours, $\frac{1}{9}tL$ burns leaving $L - \frac{1}{9}tL$ of the candle.

   We want to know the time when the length of the slower burning candle is twice the length of the faster burning candle.

$$
\begin{array}{rrcl}
 & L - \frac{1}{9}tL & = & 2 \times (L - \frac{1}{6}tL) \\
 & L - \frac{1}{9}tL & = & 2L - \frac{1}{3}tL \\
\text{Dividing by } L \text{ since } L > 0, & 1 - \frac{1}{9}t & = & 2 - \frac{1}{3}t \\
\text{Multiplying by 9,} & 9 - t & = & 18 - 3t \\
 & 2t & = & 9 \\
 & t & = & 4.5 \text{ h}
\end{array}
$$

   ∴ the slower burning candle will be double the height of the faster burning candle in 4.5 hours. Since they started burning at noon, this will occur at 4:30 p.m.

6. An airplane flies at a rate of 400 $\frac{\text{km}}{\text{h}}$ in still air. It can cover 900 km flying with the wind in the same time it takes to fly 700 km against the same wind. What is the speed of the wind?

   Let w represent the wind speed, in km/h. The speed of the plane in still air is given to be 400 km/h. Then the speed of the plane with the wind is $(400 + w)$ km/h and the speed of the plane into the wind is $(400 - w)$ km/h.

   Let t represent the time, in hours, taken to fly 900 km with the wind and 700 km against the wind.

   Using $distance = speed \times time$, we are able to come up with two equations. With the wind, we obtain $(400 + w)t = 900$ and against the wind, we obtain $(400 - w)t = 700$.

   By simplifying each equation we get $400t + wt = 900$ and $400t - wt = 700$. Adding the two equations results in $800t = 1600$ and $t = 2$ follows. The speed of the wind can be found by substituting $t = 2$ into $400t + wt = 900$ obtaining $800 + 2w = 900$ and $w = 50$ follows.

   The speed of the wind is 50 km/h.

7. Two people are running laps around a 400 m track. They begin at the same point and run in the same direction. The faster person runs at a pace of 1 kilometer every 4 minutes and the slower person runs at a pace of 1 kilometer every 6 minutes. How long will it take until the faster person laps the slower person?

   Let t represent the time, in minutes, when the faster person laps the slower person. Let d_1 represent the distance, in metres, run by the faster person and d_2 represent the distance, in metres, run by the slower person.

   At time t we want $d_1 = d_2 + 400$. (1)

   Using $distance = speed \times time$, $d_1 = \left(\frac{1000 \text{ m}}{4 \text{ min}}\right)(t \text{ min})$ and $d_2 = \left(\frac{1000 \text{ m}}{6 \text{ min}}\right)(t \text{ min})$.

   Simplifying, $d_1 = 250t$ and $d_2 = \frac{500}{3}t$. We can now substitute into (1) to solve for t.

$$\begin{aligned} 250t = &= \frac{500}{3}t + 400 \\ 750t &= 500t + 1200 \\ 250t &= 1200 \\ t &= \frac{1200}{250} = \frac{24}{5} = 4.8 \end{aligned}$$

   $\therefore$ in 4.8 minutes or 4 minutes 48 seconds the faster runner will lap the slower runner.

   The question did not ask to solve for how far each runner would run at the point that the one lapped the other. However, when the faster runner laps the slower runner, the faster runner would have ran 1200 m and the slower runner would have ran 800 m. This is found by substituting into the equations for d_1 and d_2.

8. James took a trip, first travelling on a train at 80 $\frac{\text{km}}{\text{h}}$ and then in a car at 90 $\frac{\text{km}}{\text{h}}$. The entire trip of 265 km took 3 hours to complete. If he didn't spend any time waiting between the train ride and the car ride, how long did he spend in each of the two vehicles?

   Let t be the time, in hours, travelled by train and $(3 - t)$ represent the time, in hours, travelled by car. Using $distance = speed \times time$,

$$\begin{aligned}
\text{Distance by Train} + \text{Distance by Car} &= \text{Total Distance} \\
80t + 90(3 - t) &= 265 \\
80t + 270 - 90t &= 265 \\
-10t &= -5 \\
t &= 0.5
\end{aligned}$$

   Since $t = 0.5$, $3 - t = 2.5$. Therefore he spent 0.5 hours in the train and 2.5 hours in the car.

   We can confirm the result by calculating the distances.
   By train, James travelled $80t = 80 \times 0.5 = 40$ km.
   By car, he travelled $90(3 - t) = 90 \times 2.5 = 225$ km.
   In total, he travelled $40 + 225 = 265$ km. This verifies our answer.

9. Two horsemen spot each other from 400 m apart, and start riding towards each other, one at 2 $\frac{\text{m}}{\text{s}}$ and the other at 3 $\frac{\text{m}}{\text{s}}$. A fly starts at one horse and, flying at 8 $\frac{\text{m}}{\text{s}}$, flies to the other horse, turns around and immediately flies back. If the fly continues flying back and forth until the horses meet, what total distance does the fly cover?

   Let t represent the time, in seconds, travelled by the two horsemen until they meet.
   Since the first horseman is travelling at 2 m/s, he rides $2t$ m. Since the second horseman is travelling at 3 m/s, he rides $3t$ m. Their total distance is 400 m so $2t + 3t = 400$, $5t = 400$ and $t = 80$ seconds.

   Since the fly is travelling at 8 m/s, in t seconds it travels $8t$ m. But the fly travels constantly back and forth from one rider to the other until they meet. Therefore the fly is on the move for 80 s. The total distance is $8t = 8(80) = 640$ m.

   $\therefore$ the fly covers 640 m going back and forth from rider to rider.